Name: **May 7, 2011**

FINAL

Math 5330 Abstract Algebra Online

This Final is worth 200 points. You are not allowed to use any books or notes. The last two problems are optional and each is worth 20 extra points.
You have three hours to complete the test.

1. Say what a group $(G, \cdot)$ is and give two examples for groups, where one is commutative and the other is not.

2. (a) Define that H is a subgroup of G.
   (b) Define that N is a normal subgroup of G.

3. Prove that for the additive group $(\mathbb{Z}, +)$ of integers every subgroup is of the form $k\mathbb{Z}$.

4. Let G be a finite group of order n and let H be a subgroup of order m. Outline a proof that $m|n$.

5. Let H and K be both subgroups of the group G. Assume that the orders of H and K are relatively prime. Prove that $H \cap K = \{e\}$.

6. Let N be a normal subgroup of the group G. How is the factor group G/N defined? Say what its elements are and how multiplication is defined. What is the unit of the group G/N?

7. (a) Define that ϕ is a homomorphism from the group G to the group H. How are isomorphisms defined?
   (b) Let ϕ be a homomorphism from the group G to the group H. How is the kernel N of ϕ defined?

8. Let$G$ and H be finite groups and let $G \to H$ be a surjective homomorphism. Show that the order $|H|$ of H divides the order $|G|$ of G.

9. Let $G =< x >$ be a finite cyclic group of order n. Prove that G is isomorphic to the group $\mathbb{Z}_n$ of integers modulo n.

10. Define the direct product of two groups G and H. Give examples where the direct product $\mathbb{Z}_n \times \mathbb{Z}_m$ is cyclic, and where it is not cyclic.

11. Let a and b be elements of the principal ideal domain D. Prove that $(a, b) = \{xa + yb \mid x, y \in D\}$ is an ideal. Then if d is the generator of (a, b) prove that
    (a) $d|a$ and $d|b$;
    (b) If $c|a$ and $c|b$ then $c|d$. That is d is the greatest common divisor of a and b.

12. Let $H = \{4n+1|n \in \mathbb{N}\}$, that is, $H = \{1, 5, 9, 13, 17, 21, 25, \ldots\}$. Call a number in H *irreducible* if it is different from 1 and **not** the product of two smaller numbers which belong to H. Of course, all prime numbers that belong to H are irreducible. For example, $5, 13, 17, \ldots$ are irreducible because they are prime numbers, but also the numbers $9, 21, \ldots$ are irreducible because you cannot factor them using only numbers that belong to H. Now prove:
    (a) Every number in H is a product of irreducible numbers from H.
    (b) Give an example for a number in H where the factorization into a product of irreducible numbers is not unique.