April 2, 2011

Name:

You have **90** minutes to complete the test. You cannot use any books or notes.

1. Which of the direct products are cyclic? Explain your answers.

- **a**. $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5$. Answer: cyclic, orders are pairwise prime.
- **b**. $\mathbb{Z}_2 \times \mathbb{Z}_2$. Answer: not. Each element has order 2, none cannot generate a group of order 4. It is Klein four group$V$
- **c**. $\mathbb{Z} \times \mathbb{Z}$. Not cyclic. Any possible generator like $(-1, 1)$ cannot generate any element where one component is 0

2. Calculate the order of (8,6,4) in $\mathbb{Z}_{18} \times \mathbb{Z}_9 \times \mathbb{Z}_8$. Answer
$o(8) = \frac{18}{(8,18)} = \frac{18}{2} = 9, o(6) = \frac{9}{(6.9)} = \frac{9}{(6,9)} = \frac{9}{3} = 3, o(4) = \frac{8}{(4,8)} = \frac{8}{4} = 2$ $o(8,6,4) = \text{lcm}(9,3,2) = 18$

3. Let $f : A \to B$ and $g : B \to C$ be maps such that $g \circ f : A \to C$ is injective (i.e., one-to-one). Prove that f must be injective. Answer:
Assume that $f(x_1) = f(x_2)$. Then $g(f(x_1) = g(f(x_2))$. But then $x_1 = x_2$ because $g \circ f$ is injective. Thus f is injective.

4. Find for the function $f : \mathbb{N} \to \mathbb{N}$, where $f(n) = 2n, n = 1, 2, \ldots$, some function $g : \mathbb{N} \to \mathbb{N}$ such that $g \circ f$ is the identity on $\mathbb{N}$.
Can you find some h such that $f \circ h$ is the identity on $\mathbb{N}$?Answer: $g(m) = \frac{m}{2}$ if m is even,$g(m) = 1$ (or any number)if m is odd. Then
$gf(n) = g(2n) = n$ for any n. There is no h such that $f \circ h = id$ because this would make f surjective which is not the case.

5. Find the right cosets of the subgroup $< (1,1) >$ in $\mathbb{Z}_2 \times Z_4$ Answer:
$< (1,1) >= \{(1,1), (0,2), (1,3), (0,0)\} = H$,
$\mathbb{Z}_2 \times \mathbb{Z}_4 = \{(0,0), (0,1), (0,2), (0,3), (1,0), (1,1), (1,2), (1,3)\}, H + (0,0) = H = \{(0,0), (1$
$H + (0,1) = \{(0,1), (1,2), (0,3), (1,0)\}$. There are two cosets, the coset H for $(0,0)$and the coset $H + (0,1)$.

6. Let **G** be any group and $x \in \mathbf{G}$. Let σ be the map $\sigma : y \mapsto xyx^{-1}$. Prove that this map is bijective. Answer: Assume that
$\sigma(y_1) = \sigma(y_2)$ then $xy_1x^{-1} = xy_2x^{-1}$. This yields $y_1 = y_2$ by left and right cancellation.
Let $z \in G$. Then $z = x(x^{-1}zx)x^{-1}$ shows
that σ is surjective..

7.
- **a**. Let R be an equivalence relation on the set S, and let $s \in S$. How is the equivalence class of s under R defined? Answer:
  $[s] = \{t | sRt\}$
- **b**. Let R be the equivalence relation on the set $\mathbb{R}$ of real numbers where $r \sim s$ iff $|r| = |s|$. What is the equivalence class of r?
  Answer: $[r] = \{r, -r\}$

8. **a**. Find the order of the following permutation in S_{10}:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & 4 & 5 & 2 & 1 & 7 & 8 & 6 & 10 & 9 \end{pmatrix}$$

$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & 4 & 5 & 2 & 1 & 7 & 8 & 6 & 10 & 9 \end{pmatrix} = (1,3,5)(2,4)(6,7,8)(9,1), \text{lcm}(3,2) = 6.$ The order is 6.

b. Is this permutation even or odd? Answer: This permutation is even as a product of even permutation.

9. Let p be a prime and G a group whose order is p. Prove that G is cyclic. Answer: The order of any element x different from e in G
is a divisor of the order of G which is a prime p. Thus it is p.

10. Let G be a group and let H and K be subgroups of G where $|H|$ and $|K|$ are relatively prime. Prove that $H \cap K = \{e\}$. Answer: Let
$x \in H \cap K$. Then $o(x)$ divides $o(H) = |H|$ as well as $o(x)$ divides $o(K) = |K|$. But $|H|$ and $|K|$ are relatively prime. Thus $o(x) = 1$ which is the same as $x = e$