February 14, 2015

Name:

You have **90** minutes to complete the test. You cannot use any books or notes.

1. State the Well-Ordering Principle for $\mathbb{Z}^+$. Answer: For the test, study first and second form of induction.

2. Prove by induction that $1+3+5+\cdots+(2n+1) = (n+1)^2$. **Answer**: This is obvious for $n = 1 : 1+3 = 4 = 2^2$. Assume the claim for n Then:
$1+3+5+\cdots+(2n+1)+(2(n+1)+1) = (n+1)^2+2n+3 = n^2+2n+1+2n+3 = n^2$

3. Prove by Mathematical Induction that if a set S has n elements then S has 2^n –many subsets. Answer: If S has no element, then there is only one subset, namely the empty set. Assume the claim for any set S with n –elements. Add any element c to the set S. Then the subsets of $S \cup \{c\}$ are the subsets A of S plus all sets $A \cup \{c\}$. Thus $S \cup \{c\}$ has twice as many subsets as S has. S has by induction hypothesis 2^n many subsets thus $S \cup \{c\}$ has $2 \times 2^n = 2^{n+1}$ –many subsets.

4. Is division a commutative operation on the set $\mathbb{R}^+$ of positive numbers? Is it associative? Explain your answers. Answer: of course, division is not commutative, $1/2 \neq 2/1$. It is not associative, $((1/2)/3) \neq 1/(2/3)$

5. Define that $(G,*)$ is a group. You can also state that $(G,*,^{-1},e)$ is a group. Answer: RTead this in the book.

6. Define the additive group $(\mathbb{Z}_n,\oplus)$ of integers modulo n. You have to state exactly what its elements are, how $\oplus$ is defined, what the identity is, and how the inverse of an element is defined. **Answer**: Read this in the book.

7. **a**. Find the additive inverse of 40 modulo 48. **Answer**: $-40 \bmod 48 = 8 \bmod 48$

b. Solve $15+x = 7$ modulo 48. Answer:
$x = 7-15 = -8 \bmod 48 = -8+48 = 40 \bmod 48$

8. Let $(G,*)$ be a group such that $x^2 = e$ for all $x \in G$. Show that G is abelian. Answer: $x^2 = e$ is the same as $x = x^{-1}$ for every x. Thus $xy = (xy)^{-1} = y^{-1}x^{-1} = yx$

9. Let G be a finite group, and consider the multiplication table for multiplication of G. Prove that every element of G occurs precisely once in each row and once in each column. **Answer**: This is just that given some x and z you can find some y such that $xy = z$. This is clear, $y = x^{-1}z$. This takes care for rows. For columns you argue that given any y and z you can finds some x such that $xy = z$. Here $x = y^{-1}z$.

10. Find a multiplication table on the set $A = \{a,b,c,d\}$ where every element of A occurs precisely once in each row and once in each column but where A is **not** a group. You have to explain why your algebra $A = (\{a,b,c,d\},*)$ is not a group. **Answer**: Optional puzzle. Not needed for the test.