Name: **July 7, 2015**

FINAL
Math 4377 Linear Algebra

This Final is worth 200 points. You are not allowed to use any books or notes. You have three hours to complete the Final.

1. Label the following statements as true or false.

a. An injective linear map $T : U \to V$ maps linear independent vectors to linearly independent vectors. T

b. Any set of linearly independent vectors $v_1, v_2, \dots, v_k$ in R^n can be extended to a basis by adding $n - k$ unit vectors. T

c. Any subspace of R^n is the solution-space of a linear system $Ax = 0$. T

d. Elementary row operations on an $m \times n$-matrix A don't change the space generated by the columns of A. F

e. Elementary row operations on an $m \times n$-matrix A don't change the space generated by the rows of A. T

f. A linear system $AX = 0$ of m –equations in n –unknowns has always a nontrivial solution if $n < m$ F

g. If an $m \times n$-matrix A has r-many linearly independent rows than it also must have $n - r$ many linearly independent columns. F

h. The determinant function det is a linear function on the vector space of $n \times n$ –matrices. F

i. Let A be an $n \times n$ –matrix. Then $\det(cA) = c\det(A)$. F

j. For any $n \times n$ –matrix A one has that $\det(A^t) = -\det(A)$ (A^t is the transpose of) F

a. Define that the vectors $\alpha_1, \alpha_2, \dots, \alpha_k$ are linearly independent. **Answer**: If $c_1\alpha_1 + \ldots + c_k\alpha_k = 0$ then $c = \ldots = c_k = 0$

b. Prove that $\alpha_1, \alpha_2, \dots, \alpha_k$ are linearly independent if $\alpha_1 \neq 0$ and for every $0 < i \leq k$ one has that $\alpha_i \notin < \alpha_1, \dots, \alpha_{i-1} >$Answer: By induction. $\alpha_1 \neq 0$ is linearly independent. Assume that $\alpha_1, \dots, \alpha_{i-1}$ are linearly independent. Then assume that $c_1\alpha_1 + \ldots + c_{i-1}\alpha_{i-1} + c_i\alpha_i = 0$. If we have $c_i = 0$ then $c_1\alpha_1 + \ldots + c_{i-1}\alpha_{i-1} = 0$ and by assumption $c_1 = \ldots = c_{i-1} = 0$. Thus all $c_j = 0$. But $c_i \neq 0$ yields $\alpha_i = (-c_1/c_i)\alpha_1 - \ldots - (c_{i-1}/c_i)\alpha_{i-1}$ that is $\alpha_i \notin < \alpha_1, \dots, \alpha_{i-1} >$ which is a contradiction.

2. Find a linear system with real coefficients for which the span of

$$\alpha_1 = (1,0,1,0,1), \alpha_2 = (1,0,1,1,0), \alpha_3 = (2,0,1,1,0)$$

is the solution space. **Solution**: $A = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 2 & 0 & 1 & 1 & 0 \end{pmatrix}$, **row echelon form**:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}, x_1 = 0x_2, x_3 = -x_{5,} x_4 = x_5,$$ **this gives the basis**

$X_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, X_5 = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 1 \end{pmatrix}$ **as basis for the solution space of** $AX = 0$ **and the**

matrix for the linear system for $_i$**the three vectors is** $B = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 \end{pmatrix}$

standing for the equations $x_2 = 0, -x_3 + x_4 + x_5 = 0$

3. Define that **A** is the matrix for the linear map $T : U \to V$ with respect to bases $\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_m$ of U and V, respectively.
4. Let $T : \mathsf{R}^4 \to \mathsf{R}^3$ be the linear map such that

   $T(1,0,0,0) = (1,2,3), T(0,1,0,0) = (2,6,10), T(0,0,1,0) = (0,-3,-6), T(0,0,0,1) = (-1,-3,-5)$

   a. Find the matrix of T with respect to the unit vectors.
   b. Find a basis for (T).
   c. Find a basis for $\ker(T)$.
   d. Find $T(1,1,1,1)$.

Solution: $A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ 2 & 6 & -3 & -3 \\ 3 & 10 & -6 & -5 \end{pmatrix}$, **row echelon form**: $\begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}$ **the**

rank of A **is** $2. im(T) =< \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 6 \\ 10 \end{pmatrix} >$ **The columns of the matrix** A **span**

the image! Its dimension is the column=row rank of A **and this rank is two. Any two columns of** A **span the image.**

The kernel is the solution space of $AX = 0$. **From the row-echelon form we get** $x_1 = -3x_3, x_2 = \frac{3}{2}x_3 + \frac{1}{2}x_4$**and a basis is given by**

$$X_3 == \begin{pmatrix} -3 \\ \frac{3}{2} \\ 1 \\ 0 \end{pmatrix}, X_4 = \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 1 \end{pmatrix}$$

$$T\begin{pmatrix}1\\1\\1\\1\end{pmatrix}=\begin{pmatrix}2\\2\\2\end{pmatrix}$$

5. **a.** Define that $T : \mathbf{U} \to \mathbf{V}$ is a linear map from the vector space **U** to the vector space **V**.

b. How are null-space and range of a linear map defined?

c. Let T be a linear map from R^n to R^m. Define the matrix **A** for T with respect to the unit vectors.

d. Express null space and range of T in terms of the matrix **A** for T. In particular relate column and row rank of **A** to the dimensions of the null space and the range of T. **Answers are all in the book! Recall that for $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ the matrix for T is an $n \times m$ –matrix A whose columns are the images of the unit vectors of $\mathbb{R}^n$. Thus the columns of A are** $T(e_1^n) = \begin{pmatrix}a_{11}\\a_{21}\\\cdots\\a_{n1}\end{pmatrix}$ .....,$T(e_n^n) = \begin{pmatrix}a_{1n}\\a_{2n}\\\cdots\\a_{nn}\end{pmatrix}$. **the null space of T is the solution space of $AX = 0$ it has dimension $n - r$ if r is the row-rank of A. But row-rank=column rank**=dim($range(T)$)

6. Assume that the linear map T on R^3 has matrix

$$\mathbf{A} = \begin{pmatrix}2&1&5\\3&1&7\\1&3&5\end{pmatrix}$$

a. Find a basis for the null space of T.

b. Find a basis for the range U of T.

c. Find a matrix B such that $BX = 0$ has U as solution space. **Solution**: $\begin{pmatrix}2&1&5\\3&1&7\\1&3&5\end{pmatrix}$,

row echelon form: $\begin{pmatrix}1&0&2\\0&1&1\\0&0&0\end{pmatrix}$, $x_1 = -2x_3, x_2 = -x_3$ **The null space is of dimension** 1 **and has basis** $X_3 = \begin{pmatrix}-2\\-1\\1\end{pmatrix}$

By the dimension equality: $\dim(\ker(T) + \dim(range(T) = 3.$ **Thus** $rang(T)$ **has dimension** 2 **and any two columns of** A **span the range of**

$T : range(T) =< \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} >$ **Now we need to find the matrix** B **such that**

$B \cdot \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = 0$ **and** $B \cdot \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = 0$ **We solve**

$\begin{pmatrix} 2 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0, \begin{pmatrix} 2 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}$ **has row-echelon form** $\begin{pmatrix} 2 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}$,

row echelon form: $\begin{pmatrix} 1 & 0 & 8 \\ 0 & 1 & -5 \end{pmatrix}$, **that is** $x_1 = -8x_3, x_2 = 5x_3$ **or** $X_3 = \begin{pmatrix} -8 \\ 5 \\ 1 \end{pmatrix}$

or $B = (-8, 5.1)$ **and the only one equation for the range of** T **is**
$-8x_1 + 5x_2 + x_3 = 0$

7. **Find** the inverse of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

, inverse: $\frac{1}{A} = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$

a. List the defining properties of the determinant function. **Answer**: 1) T**he determinant as function of the rows is 1**) n **–linear as function of the rows**, **2**)**alternating**, **interchanging the order of two rows then the determinant changes its sign 3**)**normed**, **the determinant of the identity is** 1

b. State the Leibnitz formula for determinants and describe how to get for an $n \times n$-matrix $n!$-many terms. Explain how for a triangular matrix only one term can be non-zero. **Wont be on the final**

8. State and prove Cramer's rule. **Just did yesterday**