Name: **July 7, 2015**

FINAL

Math 4377 Linear Algebra

This Final is worth 200 points. You are not allowed to use any books or notes. You have three hours to complete the Final.

1. Label the following statements as true or false.
   (a) An injective linear map $T : U \to V$ maps linear independent vectors to linearly independent vectors.
   (b) Any set of linearly independent vectors $v_1, v_2, \dots, v_k$ in $\mathbb{R}^n$ can be extended to a basis by adding $n-k$ unit vectors.
   (c) Any subspace of $\mathbb{R}^n$ is the solution-space of a linear system $Ax = 0$.
   (d) Elementary row operations on an $m \times n$-matrix A don't change the space generated by the columns of A.
   (e) Elementary row operations on an $m \times n$-matrix A don't change the space generated by the rows of A.
   (f) A linear system $AX = 0$ of $m-$equations in $n-$unknowns has always a nontrivial solution if $n < m$
   (g) If an $m \times n$-matrix A has r-many linearly independent rows than it also must have $n-r$ many linearly independent columns.
   (h) The determinant function det is a linear function on the vector space of $n \times n-$matrices.
   (i) Let A be an $n \times n-$matrix. Then $\det(cA) = c\det(A)$.
   (j) For any $n \times n-$matrix A one has that $\det(A^t) = -\det(A)$ (A^t is the transpose of A.)

2. (a) Define that the vectors $\alpha_1, \alpha_2, \dots, \alpha_k$ are linearly independent.
   (b) Prove that $\alpha_1, \alpha_2, \dots, \alpha_k$ are linearly independent if $\alpha_1 \neq 0$ and for every $0 < i \leq k$ one has that $\alpha_i \notin < \alpha_1, \dots, \alpha_{i-1} >$

3. Find a linear system with real coefficients for which the span of

$$\alpha_1 = (1,0,1,0,1), \alpha_2 = (1,0,1,1,0), \alpha_3 = (2,0,1,1,0)$$

   is the solution space.

4. Define that $\mathbf{A}$ is the matrix for the linear map $T : U \to V$ with respect to bases $\alpha_1, \dots, \alpha_n; \beta_1, \dots, \beta_m$ of U and V, respectively.

5. Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear map such that

$$T(1,0,0,0) = (1,2,3), T(0,1,0,0) = (2,6,10), T(0,0,1,0) = (0,-3,-6), T(0,0,0,1) = (-1,-3,-5)$$

   (a) Find the matrix of T with respect to the unit vectors.
   (b) Find a basis for $\mathrm{im}(T)$.
   (c) Find a basis for $\ker(T)$.
   (d) Find $T(1,1,1,1)$.

6. (a) Define that $T : \mathbf{U} \to \mathbf{V}$ is a linear map from the vector space $\mathbf{U}$ to the vector space $\mathbf{V}$.
   (b) How are nullspace and range of a linear map defined?
   (c) Let T be a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$. Define the matrix $\mathbf{A}$ for T with respect to the unit vectors.
   (d) Express null space and range of T in terms of the matrix $\mathbf{A}$ for T. In particular relate column and row rank of $\mathbf{A}$ to the dimensions of the null space and the range of T.

7. Assume that the linear map T on $\mathbb{R}^3$ has matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 5 \\ 3 & 1 & 7 \\ 1 & 3 & 5 \end{pmatrix}$$

   (a) Find a basis for the null space of T.
   (b) Find a basis for the range U of T.
   (c) Find a matrix B such that $BX = 0$ has U as solution space.

8. Find the inverse of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

9. (a) List the defining properties of the determinant function.
   (b) State the Leibnitz formula for determinants and describe how to get for an $n \times n$-matrix $n!$-many terms. Explain how for a triangular matrix only one term can be non-zero.

10. State and prove Cramer's rule.