Practice Test 2 with answers.

1. Which of the following subsets of R^3 are subspaces?

(a) $\{(x,y,z) \mid x+y=0$ and $y+z=0\}$

Yes. This set is the solution set of a linear system.

(b) $\{(x;y;z) \mid x=0$ or $y=0\}$

The answer is NO. The set is not closed under addition:

$$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$ **neither** $x=0$ **nor** $y=0$

(c) $\{(x;y;z) \mid x=1\}$

No. The zero vector is not in this set.

(d) $\{(x;y;z) \mid x=y$ and $z=0\}$

Yes. This is the solution set of a homogeneous linear system $x-y=0, z=0$

2. For the vector $v=(1,-1,1,-1,1) \in R^5$ find a linear system $AX=0$ whose solution space is the span of v. **Answer: First find the solution base for** $(1,-1,1,-1,1)X=0$. **This gives us four column vectors,**

$$\begin{matrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{matrix}, \quad \begin{matrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{matrix}, \quad \begin{matrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{matrix}, \quad \begin{matrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{matrix}.$$

Then the matrix A **has these as rows:**

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

3. Find a basis of the solution space of $\begin{pmatrix} 1 & -1 & 2 & 0 & 2 & 1 \\ 1 & 1 & 2 & 0 & 2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = 0.$

The rank of the matrix $\begin{pmatrix} 1 & -1 & 2 & 0 & 2 & 1 \\ 1 & 1 & 2 & 0 & 2 & 1 \end{pmatrix}$ **is obviously** $2.$ **So we should get a basis of four vectors**: $\begin{pmatrix} 1 & -1 & 2 & 0 & 2 & 1 \\ 1 & 1 & 2 & 0 & 2 & 1 \end{pmatrix}$ **has row echelon form**:

$\begin{pmatrix} 1 & 0 & 2 & 0 & 2 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$. **Notice that the coefficient of** x_4 **is** $0.$ **Don't leave it out**!

$x_4 = 1$ **leads to the second base vector**. **We get nullspace basis**:

$$\begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

4. For the vectors $\alpha_1 = (1,0,2,2)$, $\alpha_2 = (1,1,0,3)$, $\alpha_3 = (1,4,0,0)$ find a linear homogeneous system which has the span of these vectors as solution space.
Answer: **We first set up the matrix** A **which has the three vectors as rows**. **Then we solve** $AX = 0$

$\begin{pmatrix} 1 & 0 & 2 & 2 \\ 1 & 1 & 0 & 3 \\ 1 & 4 & 0 & 0 \end{pmatrix}$ **has row echelon form**: $\begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$ **It corresponds to the linear system**

$x_1 = -4x_4$
$x_2 = x_4$
$x_3 = x_4$

which has solution base $\begin{pmatrix} -4 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ **and the equation we are looking for is**

$-4x_1 + x_2 + x_3 + x_4 = 0$

5. (a) Decide whether the vector $(4, 11, 12)$ is in the span of $u = (1, 2, 3)$ and $v = (2, 5, 6)$. You have to justify your answer.

$$x_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 4 \\ 11 \\ 12 \end{pmatrix}$$ **leads to the inhomogeneous linear system** $AX = B$ **with matrix**

$\begin{pmatrix} 1 & 2 & 4 \\ 2 & 5 & 11 \\ 3 & 6 & 12 \end{pmatrix}$ **which has row-echelon form** $\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$ **from which we read off the solutions**:

$x_1 = -2$ **and** $x_2 = 3$. **Indeed**:

$-2 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + 3 \begin{pmatrix} 2 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 4 \\ 11 \\ 12 \end{pmatrix}$. **Therefore**, $(4, 11, 12)$ **is in the span of** u **and** v.

(b) Find an equation for the plane generated by the vectors $u = (1, 2, 3)$ and $v = (2, 5, 6)$

Here we could use the vector product. But this wouldn't work in higher dimensions. We use the general method as explained in the practice sheet. We first solve $AX = 0$ **where** A **is the matrix which has** u **and** v **as its rows.**

$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \end{pmatrix}$ **has nullspace basis**: $\begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$. u **and** v **Then**

$$\begin{pmatrix} -3 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = -3x_1 + x_3 = 0$$

is the equation we are looking for.

6. Prove that $u = (1,0,0)$, $v = (1,1,0)$, and $r = (1,1,1)$ form a basis of R^3. Let T be the linear map where $T(u) = (2,1,3)$, $T(v) = (1,1,2)$, $T(r) = (0,0,1)$. What is $T(6,5,3)$?

Answer:

The rank of the matrix which has the three vectors u, v, r **as rows is** 3. **But we can also argue that the first vector is not** 0, **the second vector is not a multiple of the first one and the third vector is not a linear combination of the first two vectors**. **Therefore**, u, v, r **are linearly independent and form a basis of** $\mathbb{R}^3$.

According to the formula (6.1.3) in the book one has that the A **matrix for** T **is**

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^{-1}.$$

$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^{-1}$ **is easy enough to compute**. $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$

Therefore, $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^{-1}$.

Then, $T(6,5,3) = A \begin{pmatrix} 6 \\ 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 0 & -1 \\ 3 & -1 & -1 \end{pmatrix} \begin{pmatrix} 6 \\ 5 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ 10 \end{pmatrix}$.

7. The following is a bonus problem. What is the dimension of the vector space of all symmetric n x n matrices?

I leave you in suspense.