June 15, 2016

Test 1 Math4377

Each problem is worth 20 points. You cannot use any books, notes or calculators. You have 110 minutes to complete the test.

1. (a) Find the parametric equation of the line in $\mathbb{R}^3$ that goes through the points $P = (1,1,1)$ and $Q = (1,-1,2)$

(b) Find the parametric equation of the plane in $\mathbb{R}^3$ that contains the points $P = (1,0,0), Q = (0,1,0), R = (0.0,1)$.

Solution: (**a**) $X = P + t(Q-P) = (1,1,1) + t((1,-1,2)-(1,1,1)) = (1,1,1) + t(0,-2,1)$

(**b**) $X = P + t(Q-P) + s(R-P) = (1,0,0) + t((0,1,0)-(1,0,0)) + s((0,0,1)-(1,0,0)) =$
$X = (1,0,0) + t(-1,1,0) + s(-1,0,1)$

2. Label the following statements as true or false.

   (a) The empty set is a vector space. F

   (b) In any vector space $\mathbf{V}$ the empty set $\emptyset$ generates the subspace $\{0\}$. T

   (c) The zero vector is linearly independent only in the zero space $\{0\}$. F

   (d) If $a \neq b$ and $\alpha \neq 0$ then $a.\alpha \neq b.\alpha$ T

   (e) The empty set of a vectors space V is closed. F

   (f) The union of two subspaces is never a subspace. F

   (g) The intersection of two subspaces is always a subspace. T

   (h) Subsets of linearly independent sets are linearly independent. T

   (i) Any set containing the zero vector is linearly dependent. T

   (j) Any two linearly independent vectors α,β in $\mathbb{R}^3$ can be extended to a basis of $\mathbb{R}^3$. T

3. Prove that a generating set S of n vectors in $\mathbb{R}^n$ must be linearly independent.

Proof: **Let** $S = \{\alpha_1,\ldots,\alpha_n\}$. **If** S **were linearly dependent then one of the vectors would be a linear combination of the others**, **say** α_1 **in the span of** $\alpha_2,\ldots,\alpha_n$.**Therefore** $<\alpha_2,\ldots,\alpha_n> = \mathbb{R}^n$ **but by the replacement theorem the set** $\{\alpha_2,\ldots,\alpha_n\}$ **can generate at most** $n-1$ **linearly independent vectors and not a basis of** n **vectors**.

4. Prove that any set of vectors is linearly dependent if it contains the zero vector.

Proof: **If** $S = \{0\}$ **then** S **is linearly dependent because** 0 **is linearly dependent**: $1.0 = 0$. **At any rate if** $0 \in S, 0 = 0.\alpha_1 + .. + 0.\alpha_{i-1} + 1.0 + 0.\alpha_{i+1} + \ldots + 0.\alpha_n$ shows linear dependence S.

5. Prove that the polynomials $1, x-a, (x-a)^2, \ldots, (x-a)^n$ are linearly independent.

Proof: 1 **is linearly independent**,
$x-a \notin <1>, (x-a)^2 \notin <1, x-a>, (x-a)^3 \notin <1, x-a, (x-a)^2>$ **etc. The polynomial** $(x-a)^k$ **is not a linear combination of** $1, x-a, \ldots, (x-a)^{k-1}$. **A polynomial of degree** k **cannot be a linear combination of polynomials of lower degree. This gives linear independence by a theorem that a set a sequence of vectors is linear independent if no vectors is a a linear combination of the preceding ones**.

6. Find a basis of the solution space of the linear homogeneous system:

$$x + y - z = 0$$
$$x - y - z = 0$$

Solution: $\begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & -1 \end{pmatrix}$ **has Row-Echelon-Form** $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$ **and the vector**

$X = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ **is a solution of the** 1 **–dimensional solution space which is a line**.

7. Find a basis of the subspace W of all vectors $\alpha = (a_1 + a_2 + a_3 + a_4 + a_5\}$ in $\mathbb{R}^5$ where $a_4 + a_5 = 0$.

Solution: W **is the solution space of the linear equation** $x_4 + x_5 = 0$ **which has the matrix** $\begin{pmatrix} 0 & 0 & 0 & 1 & 1 \end{pmatrix}$ **which is equivalent to**

$$x_4 = 0x_1 + 0x_2 + 0x_3 - x_5$$

and we get four linearly independent solutions:$X_1 = (1,0,0,0,0), X_2 = (0,1,0,0,0), X_3 = (0,0,1,0,0), X_5 = (0,0,0,-1,1)$. W **is of dimension four so the given four vectors form a basis of** W

8. Prove that the vectors $\alpha_1 = (1,1,1), \alpha_2 = (0,1,1), \alpha_3 = (0,0,1)$ are linearly independent in $\mathbb{R}^3$. Find the unique representation of an arbitrary vector (a_1, a_2, a_3) as linear combination of $\alpha_1, \alpha_2, \alpha_3$.

Solution $x_1(1,1,1) + x_2(0,1,1) + x_3(0,0,1) = (a_1, a_2, a_3)$ **gives us** $x_1 = a_1, x_1 + x_2 = a_2,$ **thus** $x_2 = a_2 - a_1, x_1 + x_2 + x_3 = a_3$ **therefore** $a_1 + (a_2 - a_1) + x_3 = a_3$ **which is** $x_3 = a_3 - a_2$. **So** $a_1(1,1,1) + (a_2 - a_1)(0,1,1) + (a_3 - a_2)(0,0,1) = (a_1, a_2, a_3)$. **If** $(a_1, a_2, a_3) = (0,0,0)$ **then** $x_1 = x_2 = x_3 = 0$ **which shows linear independence of** $\alpha_1, \alpha_2, \alpha_3$.

9. Prove that any subset of a linearly independent set S is linearly independent.

Proof. **Let** $T \subseteq S$ **where** S **is linearly independent**. **If we had** $c_1\alpha_1 + \ldots + c_n\alpha_n = 0$ **with** $\alpha_i \in T$ **and not all** $c_i = 0$ **then a non-trivial linear combination of vectors in** S **would be zero**, **contradicting the linear independence of** S.

10 Prove that if S is linearly independent and α is not a linear combination of S then $S \cup \{a\}$ is linearly independent.

Proof. **Let** $S = \{\alpha_1, \ldots, \alpha_n\}$. **Assume** $c_1\alpha_1 + \ldots + c_n\alpha_n + c\alpha = 0$. **If c**$\neq 0$ **then** $c_1\alpha_1 + \ldots + c_n\alpha_n = 0$ **and we get** $c_1 = \ldots = c_n = 0$ **because** S **is linearly independent**. **If** $c \neq 0$ **then** $\alpha = (-c_1/c)\alpha_1 - \ldots - (c_n/c)\alpha_n$ **would show that** α **is a linear combination of** S, **which contradicts our assumption**.