June 23, 2016

Test 2 Math4377

Each problem is worth 20 points. You cannot use any books, notes or calculators. You have 110 minutes to complete the test.

1. Label the following statements as true or false. In each part, $T : U \to V$ is a map from the finite-dimensional vector space U to V.
   (a) T is linear if $T(0_U) = 0_V$.
   (b) T is linear only if $T(0_U) = 0_V$.
   (c) If T is linear then T maps a family of linearly independent vectors to a family of linearly independent vectors.
   (d) If $\dim(U) \geq \dim(V)$ then there is a surjective linear map from U onto V.
   (e) If $\dim(U) \leq \dim(V)$ then there is an injective linear map from U to V.
   (f) If $T : U \to V$ is linear and $\alpha_1, \ldots, \alpha_n$ a basis of U then $T(\alpha_1), \ldots, T(\alpha_n)$ is a basis of V.
   (g) If $T : U \to V$ is linear and $N(T) = \{0\}$ then $R(T) = V$.
   (h) If $T : U \to V$ is linear and $R(T) = V$ then $N(T) = U$.
   (i) If $S : U \to V$ and $T : U \to V$ are both linear then $S + T$ is linear.
   (j) If $\dim(U) = \dim(V)$ then every linear map $T : U \to V$ is bijective.
   (k) If $\dim(U) = \dim(V)$ then there is a linear map $T : U \to V$ that maps a basis of U to a basis of V.
   (l) There is no linear map $T : \mathbb{R}^3 \to \mathbb{R}^4$ which is surjective.

2. Let $T : \mathbb{R}^2 \to \mathbb{R}^2, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Find a basis of $N(T)$ and a basis of $R(T)$?

4. Find the matrix of the linear map $T : \mathbb{R}^3 \to \mathbb{R}$, for which $T(x, y, z) = x - y + z$. Find a basis of $N(T)$ and a basis of $R(T)$.

5. Can you find linear maps $T : \mathbb{R}^2 \to \mathbb{R}^3$ and $S : \mathbb{R}^3 \to \mathbb{R}^2$ such that $S \circ T = id_{R^2}$ where id_{R^2} is the identity map on $\mathbb{R}^2$? Can you find such maps T and S such that $T \circ S = id_{\mathbb{R}^3}$? You must prove your answers.

6. Find the general solution of the linear system:

$$x + y - z = 1$$
$$x - y - z = 1$$

7. Let $T : U \to V, S : V \to U$ be linear and $S \circ T = id_U$. Prove that $N(T) = \{0\}$ and $R(S) = U$

8. Let $T : \mathbb{R}^3 \to \mathbb{R}$ be linear. Show that there exists numbers a, b, c such that $T(x, y, z) = ax + by + cz$.

9. Let $D : P_n[x] \to P_n[x], p(x) \mapsto p'(x)$ be the map that assigns to a polynomial its derivative. What is the matrix of this map with respect to the basis $1, x, \ldots, x^n$.

10. Let $T : U \to U$ be linear. Prove that $N(T^2) \supseteq N(T)$ and $R(T^2) \subseteq R(T)$.