July 1, 2016

Test 3 Math4377

Each problem is worth 20 points. You cannot use any books, notes or calculators. You have 110 minutes to complete the test.

1. Label the following statements as true or false.
   - a. Only invertible matrices are products of elementary matrices. T
   - b. If E is an $n \times n$ elementary matrix then E is invertible. T
   - c. $T : U \longrightarrow V$ is injective if and only if the only vector α such that $T(\alpha) = 0$ is $\alpha = 0$. T
   - d. If for a linear map $T : U \to B$ the vectors $u_1, u_2, \dots u_k$ are linearly dependent $T(u_1), T(u_2), \dots, T(u_k)$ are linearly dependent. T
   - e. If for a linear map $T : U \to B$ the vectors $u_1, u_2, \dots u_k$ are linearly independent then $T(u_1), T(u_2), \dots, T(u_k)$ are linearly independent. F
   - f. If the homogeneous system $AX = 0$ of n – equations in n –unknowns has a non-trivial solution then there is some B such that $AX = B$ has no solution. T
   - g. Given $\alpha_1, \alpha_2 \in V$ and $\beta_1, \beta_2 \in W$, there exists a linear transformation $T : V \longrightarrow W$ such that $T(\alpha_1) = \beta_1$ and $T(\alpha_2) = \beta_2$. F
   - h. Let A be an $n \times m$ –matrix. The set of $B \in \mathbb{R}^m$ for which $AX = B$ has a solution is a subspace of $\mathbb{R}^m$. T
   - i. There is a linear map $T : \mathbb{R}^3 \to \mathbb{R}^3$ for which $N(T) = R(T)$F
   - j It is impossible for the product of two non-square matrices to be invertible. F

2 Find the matrix of the linear map $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that $T(e_1) = e_1, T(e_2) = e_2 + e_1, T(e_3) = e_3 + e_2$, that is $T(e_j) = e_j + e_{j-1}, j = 2, \dots, n$. Find $N(T)$ and $R(T)$.

Solution:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$N(T) = \{0\}, R(T) -= \mathbb{R}^4$

3 Find a basis of the solution space for $x_1 + x_5 = 0, x_3 + x_4 = 0$. **Solution**:

$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$ yields $x_1 = 0x_2 - x_5, x_3 = -x_4$ and basis $X_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, X_4 = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, X_5 = \Big($

4 Find the equation $ax + by + cz + dw = 0$ of the hyperplane in R^4 which is the span the following vectors $\alpha_1 = (-1, 1, 0, 0), \alpha_2 = (1, 1, 0, 0), \alpha_3 = (1, 1, 0, 1)$. **Solution**:

$$\begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} N = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

The single equation is $z = 0$

5 Find a linear system for which the span of $\alpha_1 = (1,0,1,1,1), \alpha_2 = (0,1,3,1,4), \alpha_3 = (1,1,4,2,5)$ is the solution space. **Solution**:

$$\begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 3 & 1 & 4 \\ 1 & 1 & 4 & 2 & 5 \end{pmatrix} \text{row echelon form} \begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 3 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

There are three base vectors $\left[\begin{pmatrix} -1 \\ -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -4 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right]$ which gives us three equations $-x_1 - 3x_2 + x_3 = 0, -x_1 - x_2 + x_3 = 0, -x_1 - 4x_2 + x_5 = 0$

6 Prove that the inverse $T^{-1} : U \to V$ of an invertible linear map T is linear. **Solution**: Let $\beta_1 \in V, \beta_2 \in V$ and $T(\alpha_1) = \beta_1, T(\alpha_2) = \beta_2$. Then $T^{-1}(c_1\beta_1 + c_2\beta_2) = c_1T^{-1}(\beta_1) + c_2T^{-1}(\beta_2) = c_1\alpha_1 + c_2\alpha_2$ because $T(c_1\alpha_1 + c_2\alpha_2) = c_1T(\alpha_1) + c_2T(\alpha_2) = c_1\beta_1 + c_2\beta_2$.

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- **a**. Let A be an $n \times n$ matrix for which $A^2 = 0$. Prove that A cannot be invertible. **Solution**: If A were invertible Then $A^{-1}A^2 = A = 0$, a contradiction.
- **b**. Suppose that $AB = 0$ for some non-zero matrix B. Could A be invertible? Prove your claim. **Solution**: If A were invertible then $A^{-1}(AB) = A^{-1}0 = 0$. Thus $B = 0$, a contradiction.

8 Find the general solution of the linear system:

$$x + y - z + 3w = 1$$
$$x - y - z + 2w = 1$$

The matrix of this system is $\begin{pmatrix} 1 & 1 & -1 & 3 & 1 \\ 1 & -1 & -1 & 2 & 1 \end{pmatrix}$, row echelon form: $\begin{pmatrix} 1 & 0 & -1 & \frac{5}{2} & 1 \\ 0 & 1 & 0 & \frac{1}{2} & 0 \end{pmatrix}$ which gives us the general solution:

$$X_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -\frac{5}{2} \\ -\frac{1}{2} \\ 0 \\ 1 \end{pmatrix}$$

9 Can you find linear maps $T : \mathbb{R}^2 \to \mathbb{R}^3$ and $S : \mathbb{R}^3 \to \mathbb{R}^2$ such that $S \circ T = id_{R^2}$ where id_{R^2} is the identity map on $\mathbb{R}^2$? Can you find such maps T and S such that $T \circ S = id_{\mathbb{R}^3}$? You must prove y$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$our answers. **Solution**: T has matrix $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ and S has matrix $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and we see that $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. There is no T, S such that $T \circ S = id_{\mathbb{R}^3}$ because this would yiel that $T : \mathbb{R}^2 \to \mathbb{R}^3$ would be surjective which is impossible.

10 Let V be a vector space and let $T : V \to V$ be linear. Prove that $T^2 = T_0$ (the zero map) if and only if $R(T) \subseteq N(T)$. **Solution**: $T^2 = T_0 \Longrightarrow R(T) \subseteq N(T)$. Let $\beta \in R(T)$. Then $\beta = T(\alpha)$ for some α. But then $T(\beta) = T^2(\alpha) = 0$. Thus $\beta \in N(T)$. For the other implication $R(T) \subseteq N(T) \Longrightarrow T^2 = T_0$ Let $\alpha \in V$. Then $T(\alpha) \in R(T) \subseteq N(T)$. Thus $T(\alpha) \in N(T)$ which is $T(T(\alpha)) = T^2(\alpha) = 0$