Test 2, Math3336 **October 6, 2016**

You have the full class period to complete the test. Every problem is worth 20 points.

1. Mark as true or false.

a. A function f is injective if $f(a) = f(b)$ only if $a = b$. T

b. A function f is injective if $a = b$ then $f(a) = f(b)$. F

c. A function f is injective if $f(a) \neq f(b)$ then $a \neq b$. F

d. A function f is injective if $a \neq b$ then $f(a) \neq f(b)$. T

2. Let A be a set and $\mathsf{P}(A)$ be the power set of A. Let $f : A \to P(A)$. Mark as true or false.

a. The function f cannot be surjective. T

b. The subset $C = \{a|a \in f(a)\} \notin im(f)$ F

3. Find a surjection $f : \mathbb{N} \twoheadrightarrow \mathbb{N}$ from the set $\mathbb{N}$ of natural numbers to itself which is not injective. **Answer**: $n \longmapsto \lfloor \frac{n}{2} \rfloor$

4. Use the Cantor-Bernstein Theorem in order to prove that there is a function $f : [0,1] \to [0,1]$ such that for every $y \in [0,1]$ one has that $f^{-1}(y)$ is a two-element set. Hint: $[0,1] = [0,\frac{1}{2}) \cup [\frac{1}{2},1]$ and each of the two disjoint intervals is equivalet to $[0,1]$. **Answer**: **There is a bijection** $f_1 : [0,\frac{1}{2}) \longrightarrow [0,1], f_2 : [\frac{1}{2},1] \longrightarrow [0,1]$. **Then for** $f_1 \cup f_2$ **you have for every** $y \in [0,1]$ **exactly one** $x_1 \in [0,\frac{1}{2})$ **and one** $x_2 \in [\frac{1}{2},1]$ **such that** $f(x_1) = f(x_2) = y$

5. Determine whether each of these statements are true or false.

a) $\emptyset \in \emptyset$ F b) $\emptyset \in \{\{\emptyset\}\}$ F

c) $\{\emptyset\} = \{\emptyset,\emptyset\}$ T d) $\{\emptyset\} \subseteq \{\{\emptyset,\emptyset\}\}$ T

6. Define the successor A^+ of the set A. Find the successor of

a)$A = \emptyset$ b) $A = \{\emptyset\}$ **Answer**: $\mathbf{A}^+ = A \cup \{A\}$; a) $A = \emptyset, A^+ = \emptyset \cup \{\emptyset\} = \{\emptyset\}$ b) $A = \{\emptyset\}, A^+ = \{\emptyset\} \cup \{\{\emptyset\}\} = \{\emptyset,\{\emptyset\}\}$

7. Determine whether the function $f : \mathbb{Z} \to \mathbb{Z}$ is onto if,

a) $f(m) = \lfloor \frac{n}{2} \rfloor$,T b) $f(m) = m^2 + 1$ F c) $f(m) = m - 1$ T d) $f(m) = m^3$ F

8. Let $f : A \twoheadrightarrow B$ be a surjective function from A onto B. Let C be any subset of B. Prove that $f(f^{-1}(C) = C$. **Answer**: $d \in f(f^{-1}(C)$ **iff** $d = f(a)$ **where** $a \in f^{-1}(C)$ **iff** $f(a) \in C$. **That is** $d = f(a) \in C$. **This is** $f(f^{-1}(C) \subseteq C$. **Let** $c \in C$. **Then because** f **is surjective**, $f(a) = c$ **for some** $a \in f^{-1}(C)$. **This is** $C \subseteq f(f^{-1}(C)$

9 Assume for sets A and B that the power sets are unequal, that is $P(A) \neq P(B)$. Can you conclude that $A \neq B$? You must prove your answer.

Answer: **yes**. **We know from the practice sheet that** $P(A) = P(B)$ **yields** $A = B$. **So** $A \neq B$ **yields** $P(A) \neq P(B)$. **One can also argue that** $A \neq B$ **gives us without loss of generality some** $a \in A$ **which is not in** B. **But then** $\{a\} \in P(A)$ **but** $\{a\} \notin P(B)$.

10. a)Is the set $\{\emptyset\}$ the power set of a set? b)Is $\{\emptyset, \{\{\emptyset\}\}\}$, the power set of a set? You must prove your answers. Answer: a) yes, $\{\emptyset\} = P(\emptyset)$ b) yes, $\{\emptyset, \{\{\emptyset\}\}\} = P(\{\{\emptyset\}\})$