1. Define for integers \(a\) and \(b\) the relation \(a \mid b\).  
Prove that \(1 \mid a\) and \(a \mid 0\).  
**Answer:** \(a \mid b \iff \exists c \ a \cdot c = b\) We have \(1 \mid a\) because \(1 \cdot a = a\) and \(a \mid 0\) because of \(a \cdot 0 = 0\)

2. Let \(a\) and \(b\) be integers and let \(m\) be a positive integer. Define that \(a\) is congruent to \(b\) modulo \(m\). What are the elements congruent to \(0\)? Prove that every integer \(a\) is congruent \(\mod m\) to a unique \(0 \leq r < m\).  
**Answer:** \(a = b \mod m \iff m\left| a - b \right| \iff a - b \in m\mathbb{Z} = \{mk | k \in \mathbb{Z}\}\)  
**iff in** \(a = qm + r, 0 \leq r < m, b = qm + r, 0 \leq r < m\) one has that \(r_1 = r_2\). The elements of \(m\mathbb{Z}\) are congruent to \(0 \mod m\). And \(a = qm + r, \text{i.e.,} \ a - r = qm \in m\mathbb{Z}\) shows that \(a \equiv r\) where \(r\) is the unique remainder for \(a\) if divided by \(m\)

3. Evaluate these quantities.
   a. \(13 \mod 3\)  
**Answer:** \(13 = 4 \cdot 3 + 1, \text{i.e.,} \ 13 \equiv 1 \mod 3\)
   b. \( -97 \mod 11\)  
**Answer:** \(-97 = (-9) \cdot 11 + 2, \text{i.e.,} -97 \equiv 2 \mod 11\)
   c. \(155 \mod 19\)  
**Answer:** \(155 = 8 \cdot 19 + 3, \text{i.e.,} \ 155 \equiv 3 \mod 19\)
   d. \(-221 \mod 23\)  
**Answer:** \(-221 = (-10) \cdot 23 + 9, \text{i.e.,} \ -221 \equiv 9 \mod 23\)

4. Convert the decimal expansion of each of these integers to a binary expansion.
   a. \(22\)  
**Answer:** \(22 = 11 \cdot 2 + 0, 11 = 5 \cdot 2 + 1, 5 = 2 \cdot 2 + 1, 2 = 1 \cdot 2 + 0, 1 = 0 \cdot 2 + 1, \text{i.e.,}\) \(22 = (10110)_2\)  
   **Check:** \(0 \cdot 2^0 + 1 \cdot 2^1 + 1 \cdot 2^2 + 0 \cdot 2^3 + 1 \cdot 2^4 = 2 + 4 + 16 = 22\)
   b. \(100\)  
**Answer:** \(100 = 50 \cdot 2 + 0, 50 = 25 \cdot 2 + 0, 25 = 6 \cdot 2 + 0, 6 = 3 \cdot 2 + 0, 3 = 1 \cdot 2 + 1, 1 = 0 \cdot 2\)
   c. \(60\)  
**Answer:** \(60 = 30 \cdot 2 + 0, 30 = 15 \cdot 2 + 0, 15 = 7 \cdot 2 + 1, 7 = 3 \cdot 2 + 1, 3 = 1 \cdot 2 + 1, 1 = 0 \cdot 2 + 60 = (111100)_2\)
   d. \(9\)  
**Answer:** \(9 = 4 \cdot 2 + 1, 4 = 2 \cdot 2 + 0, 2 = 1 \cdot 2 + 0, 1 = 0 \cdot 2 + 1, \text{i.e.,}\) \(9 = 1001\)
   **Remember:** For expansion of \(a\) to base \(b\) you start with \(a = q \cdot b + a_0\) and argue that you already know that \(q = a_1 + a_2 b + \ldots + a_k b^{k-1}\), therefore, \(a = a_1 b + a_2 b^2 + \ldots + a_k b^k + a_0\). You successively divide the quotients by \(b\) to find the coefficients \(a_0, a_1, \ldots, a_k\)

5. Express the greatest common divisor of each of these pairs of integers as a combination of these integers.
   a. \(10, 11\)  
**Answer:** \(11 = 1 \cdot 10 + 1, 10 = 10 \cdot 1 + 0,\) \((11, 10) = (10, 1) = (1, 0); 1 = 1 \cdot 11 - 1 \cdot 10\)
   b. \(9, 16\)  
**Answer:** \(16 = 1 \cdot 9 + 7, 9 = 1 \cdot 7 + 2, 7 = 3 \cdot 2 + 1, 2 = 2 \cdot 1 + 0, (16, 9) = (9, 7) = (7, 2) = (2, 7) = 1 \cdot 16 - 1 \cdot 9, 2 = 1 \cdot 9 - 1 \cdot 7 = 1 \cdot 9 - 1 \cdot 7 = 1 \cdot 9 - 1 \cdot 7 = 2 \cdot 9 - 1 \cdot 16, 1 = 2 \cdot 9 - 1 \cdot 16, 1 = 1\)
1 = 4 \cdot 16 - 7 \cdot 9

c. 0, 20 \textbf{Answer:} 20 = 1 \cdot 20 + 0
d. 99, 101 \textbf{Answer:}
\[101 = 1 \cdot 99 + 2, 99 = 49 \cdot 2 + 1, 2 = 2 \cdot 1 + 0, (101, 99) = (99, 2) = (49, 1) = (1, 0)\]
\[2 = 1 \cdot 101 - 1 \cdot 99, 1 = 1 \cdot 99 - 49 \cdot 2 = 1 \cdot 99 - 49(1 \cdot 101 - 1 \cdot 49) = 50 \cdot 99 - 49\]

6. Find all invertible elements and their inverses in
   \((\mathbb{Z}_{10}, \cdot)\) \textbf{Answer:} An element \([a]\) in \(\mathbb{Z}_{10}\) is invertible iff
   \((a, 10) = 1\). The elements \(a\) that are relatively prime to 10 are
   \(1, 3, 7, 9\) and we have \([1]^{-1} = [1], [3]^{-1} = [7], [7]^{-1} = [3], [9]^{-1} = [9]\)

   \(\mathbb{Z}_{11}, \cdot) \textbf{Answer:} Because 11 is prime, all numbers between 1 and 10
   are relatively prime to 11. We have

7. Solve mod 5 the linear equation \(2x + 3 = 1\) \textbf{Answer:}
   \(2x = -2 \equiv 3 \mod 5, [2]^{-1} = [3]\). Thus \(x = 9 \equiv 4 \mod 5\). Indeed,
   \(2 \cdot 4 + 3 = 11 = 1 \mod 5\)

8. Which integers leave a remainder 1 when divided by 2 and also leave a
   remainder 1 when divided by 3. \textbf{Answer:} Of course, \(x \equiv 1 \mod 2, x \equiv 1 \mod 3\)
   \textbf{has the solution} \(x = 1\) which, according to the Chinese Remainder
   \textbf{Theorem is unique mod} 2 \cdot 3 = 6.