Test 4, Math3336

November 17, 2016

You have the full class period to complete the test. Problems 1,2,3, and 5 are each worth 15 points. Problem 4 is worth 40 points. Answers for problems 1,2,3 must be given in complete sentences in order to count.

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1. State the well-ordering principle for the set of natural numbers. **Answer**: Every non-empty set of natural numbers contains a smallest element.

2.

a. State the principle of Strong Induction. **Answer**: In ordert to show that a set S of natural numbers is equal to N one needs to verify the *Basis Step:* $1 \in S$ *Inductive Step:* $n \in S$ in case that for all $i < n$ one has that $i \in S$.

b. Prove the principle of Strong Induction from the well-ordering principle. **Answer**: Assume that there is some number $n \notin S$. Then there must be a smallest number $m \notin S$. But we have by the definition of m as the smallest number not belonging to S that for all $i < m$ one has that $m \in S$. but then by the inductive step one has that $m \in S$. this is a contradiction to $m \notin S$.

3. Use Strong Induction in order to prove that every positive natural number > 1 is a product of prime numbers. **Answer**: **Let** S be the set of natural numbers that are a product of primes. *Basis Step:* $2 \in S$. *Inductive Step:* Let n be any natural number. If n is prime then $n \in S$. Otherwise n is a product of two smaller numbers a and b. But if we assume that a and b are products of primes then $n = a \cdot b$ is a product of primes: $a = p_1 p_2 \cdots p_k, b = q_1 q_2 \cdots q_l$, then $n = (p_1 p_2 \cdots p_k) \cdot (q_1 q_2 \cdots q_l)$ is a product of primes.

4. Prove by mathematical induction.

a. $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + (n-1) \cdot n = \frac{(n-1) \cdot n \cdot (n+1)}{3}$, $n \geq 2$. **Answer**.
Basis step: $n = 2, 1 \cdot 2 = 2 = \frac{(2-1) \cdot 2 \cdot (2+1)}{3} = 2$; Inductive step: Assume that $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + (n-1) \cdot n = \frac{(n-1) \cdot n \cdot (n+1)}{3}$ then
$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + (n-1) \cdot n + n \cdot (n+1) = \frac{(n-1) \cdot n \cdot (n+1)}{3} + n \cdot (n+1) = \frac{(n-1) \cdot n \cdot (n+1) + 3n \cdot (n+1)}{3}$
$= \frac{n \cdot (n+1) \cdot [(n-1)+3]}{3} = \frac{n \cdot (n+1) \cdot (n+2)}{3}$ which proves the formula for $n+1$.

b. $\sum_{j=0}^{n} (2j+1) = (n+1)^2$ Answer. Basis step: $n = 1$:
$1 + 3 = 4 = (1+1)^2 = 4$; Inductive step: Assume that $1 + 3 + 5 + \cdots + 2n + 1 = (n+1)^2$. Then
$1 + 3 + 5 + \cdots + 2n + 1 + 2(n+1) + 1 = (n+1)^2 + 2n + 3 = n^2 + 2n + 1 + 2n + 3 = n$
which proves the formual for $n+1$.

c. Prove that $n^2 - 7n + 12$ is a non-negative integer if $n \geq 3$.
Answer:Basis step: $n = 3 : 9 - 21 + 12 = 0$ Inductive step: Assume $n^2 - 7n + 12 \geq 0$ then $(n + 1)^2 - 7(n + 1) + 12 = n^2 + 2n + 1 - 7n - 7 + 12 = n^2 - 7n + 12 + 2n - 6 = (n^2 -$
because both summands are non-negative

d. Prove that $3|n^3 + 2n$ whenever n is a positve integer. Answer. Basis step $n = 1 : 3|1 + 2 = 3.$ Inductive step. Assume $3|n^3 + 2n.$ We have $(n + 1)^3 + 2(n + 1) = n^3 + 3n^2 + 3n + 1 + 2n + 2 = (n^3 + 2n) + 3n^2 + 3n + 3 = (n^3 +$
and $3|(n + 1)^3 + 2(n + 1)$ because 3 divides both summands.

5. Give a recursive definition of $n!$ for positive natural numbers. **Answer**: Basis step:$1! = 1$ Recursive Step: $n! = (n - 1)! \bullet n$