Practice sheet for Math 3336, Test 4 Fall 2016

November 11, 2016

1. State the well-ordering principle for the set of natural numbers. Answer: Read the book!

2. State the principle of Mathematical Induction and prove it from the well-ordering principle. Answer: Explained in class. Read the book!

3. Prove by mathematical induction.

   a. \( n < 2^n \) Answer: Done in class

   b. \( \sum_{j=1}^{n} \frac{1}{2^j} = \frac{2^{n+1}-1}{2^{n+1}} \) Answer: The formula is correct for \( n = 1 : \frac{1}{2} = \frac{2^1-1}{2} \); assume that \( \sum_{j=1}^{n} \frac{1}{2^j} = \frac{2^{n+1}-1}{2^{n+1}} \) holds for some \( n \). Then

      \[
      \sum_{j=1}^{n+1} \frac{1}{2^j} = \sum_{j=1}^{n} \frac{1}{2^j} + \frac{1}{2^{n+1}} = \frac{2^{n+1}-1}{2^{n+1}} + \frac{1}{2^{n+1}} = \frac{2^{n+1}-1+2^{n+1}}{2^{n+1}} = \frac{2^{n+1}+1}{2^{n+1}}.
      \]

      By induction \( \sum_{j=1}^{n} \frac{1}{2^j} = \frac{2^{n+1}-1}{2^{n+1}} \) holds for all \( n \in N \).

4. Prove by mathematical induction.

   a. Prove that \( 3 \) divides \( n^3 + 2n \) whenever \( n \) is a positive integer. Answer: For \( n = 1 \) the formula is true: \( 3|1^3 + 2 = 3 \). Assume that for some \( n \in N \) one has that \( 3|n^3 + 2n \). Claim: \( 3|(n+1)^3 + 2(n+1) \). We have that

      \[
      (n+1)^3 + 2(n+1) = n^3 + 3n^2 + 3n + 1 + 2n + 2 = (n^3 + 2n) + 3(n^3 + n + 3)
      \]

      and we see that \( 3|n^3 + 2n \) by assumption and \( 3|3(n^3 + n + 3) \) and therefore \( 3|(n+1)^3 + 2(n+1) \). Thus \( 3|n^3 + 2n \) for every \( n \in N \).

   b. Prove that \( 2|n(n+1) \) whenever \( n \) is a positive integer. Answer: The claim is true for \( n = 1 : 2|1(1+1) = 2 \); now assume the claim for some \( n \). Then we wish to show that \( 2|(n+1)(n+2) \). But

      \[
      (n+1)(n+2) = (n^2 + n) + 2(n+1) \]

      is divisible by \( 2 \) because each summand is.

   c. Prove that \( 6|n^3 - n \) whenever \( n \) is a positive integer. Answer: For \( n = 1 \) we have that \( 6|1^3 - 1 = 0 \). Now assume \( 6|n^3 - n \) holds for some \( n \in N \). We wish to show that \( 6|(n+1)^3 - (n+1) \). Now,

      \[
      (n+1)^3 - (n+1) = n^3 + 3n^2 + 3n + 1 - n - 1 = (n^3 - n) + 3(n^2 + n).
      \]

      By the previous part, \( 2|n^2 + n = n(n+1) \). So \( 6|3(n^2 + n) \) and \( 6|(n+1)^3 - 1 \).

5. Give a recursive definition of the sum \( n + m \) of non-negative natural numbers. Answer: Let \( n \) be any natural number. We wish to define for any \( m \in N \) the sum \( n + m \). We start at \( m = 0 \). Then \( n + 0 = n \); assume that we know what \( n + m \) is. Then define \( n + m^+ = (n + m)^+ \).

6. Give a recursive definition of propositional formulas in \( \neg, \land, \lor, \Rightarrow \) and prove by structural induction that for every propositional formula the number of left parentheses is the same as the number of right parentheses. Answer: Done in class. Answer: Study Example 11 in the book on page 354.