Practice sheet for Math 3336, Test 4 Fall 2016

November 11, 2016

1. State the well-ordering principle for the set of natiural numbers. Answer: Read the book!

2. State the principle of Mathematical Induction and prove it from the well-ordering principle. Answer: **Explained in class. Read the book**!

3. Prove by mathematical induction.

a. $n < 2^n$ Answer: **Done in class**

b. $\sum_{j=1}^{n} \frac{1}{2^j} = \frac{2^n-1}{2^n}$ Answer: The formula is correct for $n = 1 : \frac{1}{2} = \frac{2^1-1}{2}$; assume that $\sum_{j=1}^{n} \frac{1}{2^j} = \frac{2^n-1}{2^n}$ holds for some n. Then $\sum_{j=1}^{n+1} \frac{1}{2^j} = \sum_{j=1}^{n} \frac{1}{2^j} + \frac{1}{2^{n+1}} = \frac{2^n-1}{2^n} + \frac{1}{2^{n+1}} = \frac{2(2^n-1)+1}{2^{n+1}} = \frac{2^{n+1}-2+1}{2^{n+1}} = \frac{2^{n+1}-1}{2^{n+1}}$. By induction $\sum_{j=1}^{n} \frac{1}{2^j} = \frac{2^n-1}{2^n}$ holds for all $n \in N$.

4. Prove by mathematical induction.

a. Prove that 3 divides $n^3 + 2n$ whenever n is a positive integer. Answer: For $n = 1$ the formula is true: $3|1^3 + 2 = 3.$ Assume that for some $n \in N$ one has that $3|n^3 + 2n.$ Claim: $3|(n+1)^3 + 2(n+1).$ we have that $(n+1)^3 + 2(n+1) = n^3 + 3n^2 + 3n + 1 + 2n + 2 = (n^3 + 2n) + 3(n^2 + n + 3)$ and we see that $3|n^3 + 2n$ by assumption and $3|3(n^2 + n + 3)$ and therefore $3|(n+1)^3 + 2(n+1).$ Thus $3|n^3 + 2n$ for every $n \in N.$

b. Prove that $2|n(n+1)$ whenever n is a positive integer. Answer: The claim is true for $n = 1 : 2|1(1+1) = 2;$ now assume the claim for some $n.$ Then we wish to show that $2|(n+1)(n+2).$ But $(n+1)(n+2) = (n^2 + n) + 2(n+1)$ is divisible by 2 because each summand is.

c. Prove that $6|n^3 - n$ whenever n is a positve integer. Answer: For $n = 1$ we have that $6|1^3 - 1 = 0.$ Now assume $6|n^3 - n$ holds for some $n \in N.$ we wish to show that $6|(n+1)^3 - (n+1).$ Now, $(n+1)^3 - (n+1) = n^3 + 3n^2 + 3n + 1 - n - 1 = (n^3 - n) + 3(n^2 + n).$ By the previous part, $2|n^2 + n = n(n+1).$ So $6|3(n^2 + n)$ and $6|(n+1)^3 - 1.$

5. Give a recursive definition of the sum $n + m$ of non-negative natural numbers. **Answer**: Let n be any natural number. We wish to define for any $m \in N$ the sum $n + m.$ We start at $m = 0.$ Then $n + 0 = n;$ assume that we know what $n + m$ is. Then define $n + m^+ = (n + m)^+.$

6. Give a recursive definition of propositional formulas in $\neg, \wedge, \vee, \Rightarrow$ and prove by structural induction that for every propositional formula the number of left parentheses is the same as the number of right parentheses. **Answer**: Done in class. **Answer**: **Study Example 11 in the book on page 354**.