June 23, 2016

Test 2 Math4377

Each problem is worth 20 points. You cannot use any books, notes or calculators. You have 110 minutes to complete the test.

1. Label the following statements as true or false. In each part, $T : U \to V$ is a map from the finite-dimensional vector space U to V.

(a) T is linear if $T(0_U) = 0_V$. F

(b) T is linear only if $T(0_U) = 0_V$. T

(c) If T is linear then T maps a family of linearly independents vectors to a family of linearly independent vectors. F

(d) If $\dim(U) \geq \dim(V)$ then there is a surjective linear map from U onto V. T

(c) If $\dim(U) \leq \dim(V)$ then there is an injective linear map from U to V. T

(d) If $T : U \to V$ is linear and $\alpha_1, \ldots, \alpha_n$ a basis of U then $T(\alpha_1), \ldots, T(\alpha_n)$ is a basis of V. F

(e) If $T : U \to V$ is linear and $N(T) = \{0\}$ then $R(T) = V$. F

(f) If $T : U \to V$ is linear and $R(T) = V$ then $N(T) = U$. F

(g) If $S : U \to V$ and $T : U \to V$ are both linear then $S + T$ is linear. T

(h) If $\dim(U) = \dim(V)$ then every linear map $T : U \to V$ is bijective. F

(i) If $\dim(U) = \dim(V)$ then there is a linear map $T : U \to V$ that maps a basis of U to a basis of V. T

(j) There is no linear map $T : \mathbb{R}^3 \to \mathbb{R}^4$ which is surjective. T

2. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$, $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Find a basis of $N(T)$ and a basis of $R(T)$?

Solution. We have $\dim R(T) = 1$ **and** $\dim(N(T) = 1$ **and** $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ **is a basis of** $R(T)$ **and** $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ **a basis of** $N(T)$.

4. Find the matrix of the linear map $T : \mathbb{R}^3 \to \mathbb{R}$, for which $T(x,y,z) = x - y + z$. Find a basis of $N(T)$ and a basis of $R(T)$.

Solution: $Mat(T) = (1,-1,1)$; **a basis of** $N(T)$ **is given by a basis of** $x - y + z = 0$ **which is** $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ **and** $R(T) = \mathbb{R}$ **and a basis of the reals is** 1.

5. Can you find linear maps $T : \mathbb{R}^2 \to \mathbb{R}^3$ and $S : \mathbb{R}^3 \to \mathbb{R}^2$ such that $S \circ T = id_{R^2}$ where id_{R^2} is the identity map on $\mathbb{R}^2$? Can you find such maps T and S such that $T \circ S = id_{\mathbb{R}^3}$?

You

must prove your answers.

Solution. **Define linear maps**

$T : \mathbb{R}^2 \to \mathbb{R}^3, e_1^2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = e_1^3, e_2^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = e_2^3$ **and**

$S : \mathbb{R}^3 \longrightarrow \mathbb{R}^2, e_1^3 \mapsto e_1^2, e_2^3 \mapsto e_2^2, e_3^3 \mapsto \alpha$ **any vector in** $\mathbb{R}^2$.**Then** $S \circ T$ **is the identity on the basis** $\{e_1^2, e_2^3\}$ **and therfore the identity on** $\mathbb{R}^2$.

There cannot be maps T **and** S **such that** $T \circ S = id_{\mathbb{R}^3}$. **For any map** $S : \mathbb{R}^3 \to \mathbb{R}^2$ **we have** $R(S)$ **has as a subspace of** $\mathbb{R}^2$**that** $\dim R(S) \leq 2$.

Thus $\dim N(S) \geq 1$ **by the dimension equality** $\dim N(S) + \dim R(S) = 3$. **if there is a vector** β **in** $\mathbb{R}^3$ **such that** $S(\beta) = 0$ **then** $(T \circ S)(\beta) = 0$

and cannot be the identity on $\mathbb{R}^3$.

6. Find the general solution of the linear system:

$$x + y - z = 1$$
$$x - y - z = 1$$

Solution. $A = \begin{pmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$ is the matrix of this linear sytem. It has the row echelon form: $\begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$, which

gives the equations $x = 1 + z, y = 0 + 0z$.All solutions are given by

$X = \backslash \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + z \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$which is a line.

7. Let $T : U \to V, S : V \to U$ be linear and $S \circ T = id_U$.Prove that $N(T) = \{0\}$ and $R(S) = U$

Solution: T **must be injective**, **therefore** $N(T) = \{0\}$. **Or**, **if we had some** $\alpha \neq 0$ **such that** $T(\alpha) = 0$ **then** $S(T(\alpha)) = \alpha$ **and** $S(T(\alpha)) = S(0) = 0$, **a contradiction**. **Let** $\alpha \in U$ **arbitrarily chosen**. **Then** $\alpha = S(T(\alpha)) = S(\beta)$, **thus** $\alpha \in R(S)$.

8. Let $T : \mathbb{R}^3 \to \mathbb{R}$ be linear. Show that there exists numbers a, b, c such that

$T(x, y, z) = ax + by + cz.$

Solution: **The matrix for** T **is a** $1x3$ **–matrix**. $A = (a, b, c)$ **where**

$a = T(1,0,0), b = T(0,1,0), c = T(0,0,1)$ **and** $T(x,y,z) = (a,b,c)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = ax + by + cz.$

9. Let $D : P_n[x] \to P_n[x], p(x) \longmapsto p'(x)$ be the map that assigns to a polynomial its derivative.

What is the matrix of this map with respect to the basis $1, x, \ldots, x^n$.

Solution: **The matrix is** $(n+1)x(n=1)$. **say for** $n = 3$ $P_3[x]$ **has a basis** $1, x, x^2, x^3$ **and the matrix of the differentiation operator is**

$$Mat(D) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

10. Let $T : U \to U$ be linear. Prove that $N(T^2) \supseteq N(T)$ and $R(T^2) \subseteq R(T)$.

Solution: **If** $\alpha \in N(T)$ **then** $T(\alpha) = 0$ **but then** $T^2(\alpha) = T(T(\alpha)) \ = T(0) = 0.$ **Therefore** $\alpha \in N(T^2)$, **which is** $N(T) \subseteq N(T^2)$.

Let $\alpha \in R(T^2)$. **That is** $\alpha = T^2(\beta)$ **for some** $\beta \in U$. **But then** $\alpha = T(\gamma)$ **for** $\gamma = T(\beta)$ **which shows** $\alpha \in R(T)$, **which is** $R^2(T) \subseteq R(T)$.