June 22, 2017

Test 2 Math4377

Each problem is worth 20 points. You cannot use any books, notes or calculators. You have 110 minutes to complete the test.

1. Label the following statements as true or false. In each part, $T : U \to V$ is a map from the finite-dimensional vector space U to V.

(a) T is linear if $T(0_U) = 0_V$. F

(b) T is linear only if $T(0_U) = 0_V$. T

(c) If T is linear then T maps a family of linearly dependent vectors to a family of linearly independent vectors. F

(d) If $\dim(V) \geq \dim(U)$ then there is a surjective linear map from V onto U. T

(e) If $\dim(U) \leq \dim(V)$ then there is an injective linear map from U to V. T

(f) If $T : U \to V$ is surjective linear and $\alpha_1, \ldots, \alpha_n$ a basis of U then $T(\alpha_1), \ldots, T(\alpha_n)$ is a basis of V. F

(g) If $T : U \to V$ is linear and $N(T) = \{0\}$ then $R(T) = V$. F

(h) If $T : U \to V$ is linear and $R(T) = V$ then $N(T) = U$. F

(i) If $T : U \to V$ and $S : V \to W$ are both linear then $S \circ T$ is linear. T

(j) If $\dim(U) = \dim(V)$ then every linear map $T : U \to V$ is bijective. F

(k) If $\dim(U) = \dim(V)$ then there is a linear map $T : U \to V$ that maps a basis of U to a basis of V. T

(l) There is a linear map $T : \mathbb{R}^3 \to \mathbb{R}^4$ which is surjective. F

2. Can you find a linear map $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ for which $N(T) = R(T)$? You must explain your answer. **Solution**: **No, according to the dimension equality we have** $n(T) + r(T) = 3$ **and we cannot have** $N(T) = R(T)$ **which would give us** $n(T) = r(T)$ **and therefore** $2 \cdot n(T) = 3$

3. Let $S : U \to V$ and $T : U \to V$ be linear. Prove that $T + S : U \to V$ is linear where $(T + S)(\alpha) = T(\alpha) + S(\alpha)$ **Solution**: **We need to show 1)** $(T + S)(\alpha + \beta) = (T + S)(\alpha) + (T + S)(\beta)$, **2)** $(T + S)(c.\alpha) = c.(T + S)(\alpha)$. **ad1)**: $(T + S)(\alpha + \beta) = T(\alpha + \beta) + S(\alpha + \beta) =$
$= T(\alpha) + T(\beta) + S(\alpha) + S(\beta) = T(\alpha) + S(\alpha) + T(\beta) + S(\beta) = (T + S)(\alpha) + (T + S)(\beta)$. **ad 2**: $(T + S)(c.\alpha) = T(c.\alpha) + S(c\alpha) = c.T(\alpha) + c.S(\alpha) = c.(T(\alpha) + S(\alpha)) = c.(T + S)(\alpha)$.

4. Find the matrix of the linear map $T : \mathbb{R}^3 \to \mathbb{R}$, for which $T(x, y, z) = x + y + z$. Find a basis of $N(T)$ and a basis of $R(T)$.

Solution: $Mat(T, \varepsilon_1^3, \varepsilon_2^3, \varepsilon_3^3; 1) = (1, 1, 1)$; **to find a basis of** $N(T)$ **we need to solve** $x + y + z = 0, x = -y - z$. **A solution base is given by** $X_y = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ **and** $X_z = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$, **it is a two-dimensional subspace of** $\mathbb{R}^3$ **and a basis of** $N(T)$; $R(T) = \mathbb{R}$.

5. Can you find linear maps $T : \mathbb{R}^2 \to \mathbb{R}^3$ and $S : \mathbb{R}^3 \to \mathbb{R}^2$ such that $S \circ T = id_{R^2}$ where id_{R^2} is the identity map on $\mathbb{R}^2$? Can you find such maps T and S such that $T \circ S = id_{\mathbb{R}^3}$? You must prove your answers. **Solution**: **We take the linear maps where** $T(\varepsilon_1^2) = \varepsilon_1^3, T(\varepsilon_2^3) = \varepsilon_2^3; S(\varepsilon_1^3) = \varepsilon_1^2, S(\varepsilon_2^3) = \varepsilon_2^2, S(\varepsilon_3^3) = \begin{pmatrix} 100 \\ 13 \end{pmatrix}$ **or anything**.

$Mat(T) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, Mat(S) = \begin{pmatrix} 1 & 0 & 100 \\ 0 & 1 & 13 \end{pmatrix}$

$\begin{pmatrix} 1 & 0 & 100 \\ 0 & 1 & 13 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1.0 & 0 \\ 0 & 1.0 \end{pmatrix}$ **You cannot find such maps T and S such that $T \circ S = id_{\mathbb{R}^3}$. This would make S injective and there is no injective map from $\mathbb{R}^3$ to $\mathbb{R}^2$.**

6. Find the general solution of the linear system:

$$x + y + z + u = 1$$
$$x - z = 1$$

Solution: $\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 0 & 1 \end{pmatrix}$, **row echelon form**: $\begin{pmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 2 & 1 & 0 \end{pmatrix}$ **stands for** $x = 1 + z, y = -2z - u$ **Particular Solution** $X_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ **basis of the homogeneous system** $X_z = \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, X_u = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$ **The general solution is** $X_0 + z.X_z + u.X_u$

7. Let $T : U \to V, S : V \to U$ be linear and $S \circ T = id_U$. Prove that T is injective and S is surjective. **Solution**: **Assume** $T(\alpha) = 0$ **then** $S(T(\alpha)) = S(0) = 0$ **but** $S(T(\alpha)) = \alpha$ **therefore** $\alpha = 0$. **thus T is injective**. **let** $\alpha \in U$ **Then** $S(T(\alpha)) = \alpha$ **and therefore** $S(\beta) = a$ **for** $\beta = T(\alpha)$. **Thus S is surjective**.

8. Let $T : \mathbb{R}^3 \to \mathbb{R}, T(x,y,z) = ax + by + cz$. What are $T(\varepsilon_1^3), T(\varepsilon_2^3), T(\varepsilon_3^3)$?. What is $Mat(T; \varepsilon_1^3, \varepsilon_2^3, \varepsilon_3^3; \varepsilon_1^1 = 1)$ S**olution**: $T(\varepsilon_1^3) = a, T(\varepsilon_2^3) = b, T(\varepsilon_3^3) = c$ **We have** $Mat(T; \varepsilon_1^3, \varepsilon_2^3, \varepsilon_3^3; \varepsilon_1^1 = 1) = (a, b, c)$

9. Let $D : P_n[x] \to P_n[x], p(x) \mapsto p'(x)$ be the map that assigns to a polynomial its derivative.

What is the matrix of this map with respect to the basis $1, x, \ldots, x^n$. What is $N(D)$

and $R(D)$? **Solution**: $Mat(D)$ **is an** $(n+1) \times (n+1)$ –**matrix**

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & 0 & n-1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$N(D)$ **is the set of constant polynomials** $c \in \mathbb{R}$. **It is a one-dimensional space**; $R(D)$ **is the set of polynomials of degree** $\leq n-1$: **Any such polynomial** $a_0 + a_1 x + \ldots + a_{n-1}x^{n-1}$ **is the derivative of a polynomial in** $P_n[x]$.

10. Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}.$$

Prove that A is invertible and find the inverse of A. Solution:The three rows are linearly independent: the second row is not a multiple of the third row and the first row can not be a linear combination of the second and third row. We have that

$$A^{-1} = \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{12} \\ 0 & \frac{1}{4} & -\frac{5}{24} \\ 0 & 0 & \frac{1}{6} \end{pmatrix}$$