June 14, 2017

Test 1 Math4377

Each problem is worth 20 points. You cannot use any books, notes or calculators. You have 110 minutes to complete the test.

1. (a) Find the parametric equation of the line in $\mathbb{R}^3$ that goes through the points $P = (1,-1,1)$ and $Q = (1,-1,2)$

(b) Find the parametric equation of the plane in $\mathbb{R}^3$ that contains the points $P = (1,0,0), Q = (1,1,0), R = (1,1,1)$.

Solution: (**a**) $X = P + t(Q-P) = (1,-1,1) + t((1,-1,2) - (1,-1,1)) = (1,-1,1) + t(0,0,1)$

(**b**) $X = P + t(Q-P) + s(R-P) = (1,0,0) + t((1,1,0) - (1,0,0)) + s((1,1,1) - (1,0,0)) =$
$X = (1,0,0) + t(0,1,0) + s(0,1,1)$

2. Find a basis of the subspace of $\mathbb{R}^4$ which is generated by $(1,0,2,3), (1,0,3,3), (3,0,7,9), (4,0,12,15)$.

Solution: $\begin{pmatrix} 1 & 0 & 2 & 3 \\ 1 & 0 & 3 & 3 \\ 3 & 0 & 7 & 9 \\ 4 & 0 & 12 & 15 \end{pmatrix}$, row echelon form: $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, The rank is 3. The vectors $(1,0,0,0), (0,0,1,0), (0,0,0,1)$ form a basis. Amongst the 4 given vectords, the third is a linear combination of the first two vectors. Therefore $(1,0,2,3), (1,0,3,3), (4,0,12,15)$ form also a basis. .

3. Label the following statements as true or false.

(a) $\{0\}$ is a vector space. T

(b)The zero vector 0 is linearly independent. F

(c) Every subspace contains the zero vector. T

(d) If $a \neq b$ and $\alpha \neq 0$ then $a.\alpha \neq b.\alpha$ T

(e) The empty set of a vectors space V is closed. F

(f) The union of two subspaces is a subspace. F

(g) The field $\mathbb{C}$ of complex numbers is of dimension 2 as vector space over $\mathbb{R}$, T

(h) The field $\mathbb{R}$ is a vector space over the field $\mathbb{C}$ of complex numbers. F

(i) Any set not containing the zero vector is linearly independent. F

(j) Any three vectors in $\mathbb{R}^2$ are linearly dependent. T

4. Prove that In $\mathbb{R}^3$ any two non-parallel vectors α and β can be extended to a basis by adding one of the three unit vectors $\varepsilon_1, \varepsilon_2, \varepsilon_3$ to α and β.

Proof: **This follows from the replacement theorem.** $G = \{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ **is a generating set and** $I = \{\alpha, \beta\}$ **is linearly independent. Two of the three unit vectors can be replaced by** α **and** β **The resulting set** G'contains α and β and one of the three unit vectors and is generating. But any generating set of three vectors in $\mathbb{R}^3$ is linearly independent.

6. Prove that the polynomials $1, x-a, (x-a)^2, \ldots, (x-a)^n$ are linearly independent.

Proof: 1 **is linearly independent**,

$x-a \ \notin < 1 >, (x-a)^2 \ \notin < 1, x-a >, (x-a)^3 \ \notin < 1, x-a, (x-a)^2 >$ **etc. The polynomial** $(x-a)^k$ **is not a linear combination of** $1, x-a, \ldots, (x-a)^{k-1}$. **A polynomial of degree** k **cannot be a linear combination of polynomials of lower degree. This gives linear independence by a theorem that a sequence of vectors is linear independent if no vectors is a a linear combination of the preceding ones.**

7. Find a basis of the solution space of the linear homogeneous system:

$$x+y=0$$

$$x-y-z=0$$

Solution: **The matrix of the linear system is** $\begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & -1 \end{pmatrix}$, row echelon form: $\begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} \end{pmatrix}$, . This stands for the system $x = \frac{1}{2}z, y = -\frac{1}{2}z$ The solution space is of dimension 1 and a base vector is $X = (\frac{1}{2}, -\frac{1}{2}, 1)$

8. Find a basis of the subspace W of all vectors $\alpha = (a_1, a_2, a_3, a_4, a_5\}$ in $\mathbb{R}^5$ where $a_1+a_2+a_3+a_4+a_5 = 0$

Solution:

$a_1 = -a_2-a_3-a_4-a_5, X_2 = (-1,1,0.0.0), X_3 = (-1,0,1,0,0), X_4 = (1.0,0,1,0), X_5 = (-1.0$

9. Let $p(x)$ be a polynomial of degree $3.$ Prove that $p(x), p'(x), p''(x), p'''(x)$ are linearly independent.

Solution $\deg(p(x)) \ = 3, \deg(p'(x)) \ = 2, \deg(p''(x) = 1, \deg(p'''(x) = 0.$ **We have** $p'''(x) \neq 0, p''(x) \ \notin < p'''(x) >, p'(x) \ \notin < p''(x), p'''(x) >, p(x) \ \notin, < p'(x), p''(x), p'''(x) >.$ **Thus they are linearly independent.**

10. Prove that T is linearly dependent if it contains a linearly dependent subset $S.$

Solution. without loss of generality, $T = \{\alpha_1, \ldots \alpha_k, \alpha_{k+1}, \ldots \alpha_n\}$ **where** $S = \{\alpha_1, \ldots \alpha_k\}$.**We have** $c_1, \ldots c_k$ **not all** $0,$ **such that** $c_1\alpha_1 + \ldots + c_k\alpha_k = 0.$ **But then** $c_1\alpha_1 + \ldots + c_k\alpha_k + 0\alpha_{k+1} + \ldots + 0\alpha_n = 0$ **is a linear combination being where not all coefficients are zero. This shows linear dependence of** T.