Sums and Intersections of Subspaces

Let $\alpha_1, \cdots, \alpha_m$ be vectors of a space U. The span V of these vectors is the smallest subspace of U that contains these vectors. Obviously it is the set of all linear combinations of the α_i :

$$V = span\{\alpha_1, \cdots, \alpha_m\} =< \alpha_1, \cdots, \alpha_m >= \{c_1 \cdot \alpha_1 + \ldots + c_m \cdot \alpha_m | \ c_i \in F\}$$

where F is the field of scalars. In our case it is either the field $\mathbb{R}$ of reals or $\mathbb{C}$ the field complex numbers. Now, how can we find a basis of V? For this the following lemma is helpful:

Lemma.

1. $< \alpha_1, \cdots, \alpha_m >=< c \cdot \alpha_1, \alpha_2, \cdots, \alpha_m >$ where $c \in F, c \neq 0$;

2. $< \alpha_1, \cdots, \alpha_m >=< \alpha_1 + c \cdot \alpha_2, \alpha_2, \cdots, \alpha_m >$ where c is any element of F

In words:

1. The span remains unchanged if any of the spanning vectors is replaced by a non-zero multiple of it.

2. The span remains unchanged if a multiple of a spanning vector is added to another spanning vector.

The proof follows from the fact that every vector of the left-hand side is a linear combination of the spanning vectors on the right-hand side.

In particular we get the following

Theorem Let A be na $m \times n$ – matrix. Then the non-zero rows of the row-echelon form of A form a basis of the space V that is generated by the rows of A.

The union of two subspaces V and W is in general not a subspace of U. The sum of vectors v and w is in general neither in V nor W. The smallest subspace that contains V and W is called the sum $V + W$. It is just the sum of vectors from V and W.

Theorem. Let V and W be subspaces of the vector space U. Then the sum of V and W

$$V + W = \{v + w | \ v \in V, w \in W\}$$

is the smallest subspace of U that contains V and W. That is :

$$< V \cup W >= V + W$$

If $U = F^n$ and $V =< v_1, \ldots, v_s >, W =< w_1, \ldots, w_t >$ then the $s + t$-many rows of the matrix $A = \{v_1 | \ldots | v_s | w_1 | \ldots | w_t >$ form a spanning set of $V + W$. We have that

$$\dim(V + W) = rank(A)$$

and the non-zero rows of $REF(A)$ form a basis of $V + W$.

For any subspace V of $U = F^n$ we have the space of all vectors x which are perpendicular to every vector $v \in V$

$$V^\perp = \{x \in U | v \cdot x = 0, \text{ for all } v \in V\}$$

Here "$\cdot$" stands for the dot-product and we may consider the $v's$ as rows and the $x's$ as columns.

Let $V =< v_1, \ldots, v_m >$ and $A = (v_1|\cdots|v_m)$ be the $m \times n$-matrix which has the spanning vectors of V as its rows then the solution space W of the linear homogeneous system

$$Ax = 0$$

is just $V^{\perp}$. If $r = rank(A)$ then

$$\dim(V^{\perp}) = n - r$$

and from the row-echelon form of A we get a basis of $s = n - r$ many basis solutions w_i of $Ax = 0$. If $B = (w_1|\ldots|w_s)$ is the matrix which has the w_i as its rows and therefore $rank(B) = s$, then

$$Bx = 0$$

is an equational system for V. That is:

$$Bx = 0 \text{ iff } x \in V$$

This follows from $V^{\perp\perp} = V$: We have $Bx = 0$ iff $x \in W^{\perp}$. Clearly, $V \subseteq W^{\perp}$ and $\dim(V) \leq \dim(W^{\perp})$, where $\dim W^{\perp} = n - s = n - (n - r) = r = \dim(V)$.
In particular we have a method of finding an equational system $Bx = 0$ for for the sum of subspaces V and W of the space $U = F^n$.
Given subspaces V and W of U then the intersection is a subspace of U. This is quite trivial to see. But how can we find a basis of $V \cap W$? Assume again that $U = F^n$ and $V =< v_1, \ldots, v_s >$ and $W =< w_1, \ldots w_t >$. According to the previous method we find an equational system $Bx = 0$ for V and $Cx = 0$ for W. Let $A = (B|C)$ be the matrix which has as its rows the rows of B and C. That is the equations of V and W combined are the equations of $V \cap W$. Then

$$\dim(V \cap W) = n - rank(B|C)$$

Example :

Let $U = \mathbb{R}^4$ and $V =< (1,0,0,0),(0,1,0,0) >, W =< (0,0,1,0),(0,0,0,1) >$. That is V is generated by the first two unit vectors and W is generated by the third and fourth unit vector.

$$V + W = R^4$$

For U we have two equations $x_3 = 0, x_4 = 0$ which gives us the matrix $B = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. Similarly W is the solution set of the equational system $x_1 = 0, x_2 = 0$ and W has matrix $C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$. The matrix A for $V \cap W$ is

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, rank(A) = 4$$

which stands for $x_3 = 0, x_4 = 0, x_1 = 0, x_2 = 0$. Its solution space is the null space and in this case we have

$$V \cap W = 0$$

We have that the intersection of two two-dimensional subspaces is the null space. There is a general theorem

Theorem

$$\dim(V) + \dim(W) = \dim(V + W) + \dim(V \cap W)$$