November 17. 2018

Test 3 Math5331

Each problem is worth 20 points. You cannot use any books, notes or calculators. You have 110 minutes to complete the test.

1. Label the following statements as true or false.

a. Only invertible matrices are products of elementary matrices. T

b. If E is an $n \times n$ elementary matrix then E is invertible. T

c. The inverse of an elementary matrix is not always elementary. F

d. The row-echelon form of a square matrix A is the identity matrix if and only if A is invertible T

e. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be a basis of the vector space U. If $T : U \to U$ is linear then $T(\alpha_1), T(\alpha_2), \ldots, T(\alpha_n)$ is a basis of U. F

f. If the homogeneous system $AX = 0$ of n equations in n unknowns has a non-trivial solution then there is some b such that $AX = b$ has no solution. T

g. The solution set of $AX = b$ is always a subspace. F

h. If $T : U \to V$ is linear where $\dim(U) > \dim(V)$ then $\dim(ker(T)) > 0$. T

i. It is impossible for the product of two non-square matrices to be invertible. F

j. The homogeneous linear system $Ax = 0$ of n linear equations in n unknowns has a non-trivial solution if and only if A is not invertible. T

2. Assume that $T : U \to V$ is linear and $S : V \to V$ is invertible. Show that for the composition $S \circ T : U \to V$ one has that $N(T) = N(S \circ T)$.
Answer: $N(T) \subseteq N(S \circ T)$ is always true. $T(\alpha) = 0$ implies that $S(T(\alpha)) = 0.$ Now assume that $S(T(\alpha)) = 0$. Assuming that S has an inverse, then $T(\alpha) = 0$

3. Find the general solution of the system of two equations in four unknowns:

$$x_1 + x_3 + 2x_4 = 5$$

$$x_3 + 3x_4 = 7$$

Answer: $\begin{pmatrix} 1 & 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & 3 & 7 \end{pmatrix}$, row echelon form: $\begin{pmatrix} 1 & 0 & 0 & -1 & -2 \\ 0 & 0 & 1 & 3 & 7 \end{pmatrix}$ stands for $x_1 = -2 + x_4, x_3 = 7 - 3x_4.$ Particular solution if $x_2 = x_4 = 0, x_1 = -2, x_3 = 7$
$X_0 = \begin{pmatrix} -2 \\ 0 \\ 7 \\ 0 \end{pmatrix}$, basis for general solution $X_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, X_4 = \begin{pmatrix} 1 \\ 0 \\ -3 \\ 1 \end{pmatrix}$ general solotion $X = X_0 + x_2X_2 + x_{4X_4}$

4. Let T be the linear map on R^4 for which $e_1 \to e_2, e_2 \to e_3, e_3 \to e_4, e_4 \to e_1$. What is the matrix A of T with respect to the unit vectors e_i? Why is A invertible? Find the inverse of

A. **Answer**: $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, inverse: $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

5 Find the equation $ax + by + cz = 0$ of the plane in R^3 which is the span the following vectors $\alpha_1 = (1,0,1)$ and $\alpha_2 = (1,1,0)$. Answer: $\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$, nullspace basis: $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$, yields $-x + y + z = 0$ as equation which has α_1 and α_2 as solutions.

6 Prove that a linear system $Ax = 0$ of m equations in n unknowns has a solution $x \neq 0$ if $m < n$.

7 Solve the following system of linear equations

$$\begin{aligned} x_1 + 2x_2 - x_3 &= -1 \\ 2x_1 + 2x_2 + x_3 &= 1 \\ 3x_1 + 5x_2 - 2x_3 &= -1 \end{aligned}$$

Answer: $\begin{pmatrix} 1 & 2 & -1 & -1 \\ 2 & 2 & 1 & 1 \\ 3 & 5 & -2 & -1 \end{pmatrix}$, row echelon form: $\begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -1 \end{pmatrix}$,
$x_1 = 4, x_2 = -3, x_3 = -1$

8 Let A be an $n \times n$ matrix for which $A^2 = 0$. Prove that A cannot be invertible. Answer: If A had an inverse A^{-1} then $A^{-1}A^2 = A^{-1}0 = 0$, thus $A = 0$ which is not invertible.

9 Find the rank of $A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}$ and find A^{-1} if it exists. Answer: $\begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}$,
inverse: $\begin{pmatrix} \frac{1}{6} & -\frac{1}{3} & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{pmatrix}$

10 Find a linear system $AX = 0$ whose solution space is the span of $(1,-1,1,1), (1,1,0,0)$. space is the span of $(1,-1,1,1), (1,1,0,0)$. space is the span of $(1,-1,1,1), (1,1,0,0)$. Answer: $\begin{pmatrix} 1 & -1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$, nullspace basis:

$$\left[\begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 1 \end{pmatrix}\right]$$ yields equations $-x_1 + x_2 + 2x_3 = 0, -x_1 + x_2 + 2x_4 = 0$