Axioms of a vector space

A vector space is an algebraic system V consisting of a set whose elements are called *vectors* but vectors can be anything. A geometric interpretation of vectors as being directed arrows helps our understanding of the rules and laws of vector algebra, but it is not necessary: The inventory of a car repair shop can be represented as a vector whose components stand for batteries, tires alternators etc. Thus $\alpha = (20, 4, 3, 8)$ may stand for 20 tires, 4 batteries , 4 tires, 3 alternators, 8 break pedals,... This interpretation of the quadruple $(20, 4, 3, 8)$ is void of any geometric interpretation. What matters is that we can add $n-tuples$ of numbers :

$\alpha = (a_1, a_{2,\ldots,}, a_n), \beta = (b_1, b_2, \ldots, b_n)$ can be added component wise:

$\alpha + \beta = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$

and multiplied vectors by numbers (scalars):

$r.\alpha = (r \cdot a_1, r \cdot a_2, \ldots, r \cdot a_n)$

here r can be any real number. If this is the case then we speak about a vector space over the field $\mathbb{R}$ of real numbers. But we can choose any field, like rational numbers $\mathbb{Q}$ or complex numbers $\mathbb{C}$. The book deals with fields in the Appendix. In this course it is enough that *scalars,* that is numbers, are just reals.

Now, with respect to addition, vectors form what is known a *commutative group*.

Associative law: $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$

Existence of a neutral element or unit: $\alpha + 0 = 0 + \alpha = \alpha$

Additive Inverse: $\alpha + (-\alpha) = (-\alpha) + \alpha = 0$

Commutativity $\alpha + \beta = \beta + \alpha$

The binary operation is here denoted as $+$ and called addition. If the binary group operation is not commutative then the operation is called multiplication and usually denoted as $*$. We will later see that $n \times n-$ matrices can be multiplied and the product is associative. The $n \times n$ identity matrix $I_{n \times n}$ is the unit with respect to matrix multiplication and some matrices have an inverse. But multiplication of matrices is not commutative.

Addition of vectors $\alpha, \beta, \ldots$ and multiplicaton by scalars $r, s..$ are related by four more axioms or rules:

associative law: $(r \cdot s).\alpha = r.(s.a)$

distributive laws: $(r + s).\alpha = r.\alpha + s.\alpha, \qquad r.(\alpha + \beta) = r.\alpha + r.\beta$

unitary law: $1.\alpha = \alpha$

A mathematical structure V with an operation called addition and multiplication by scalars $r, s, ..$from a field F is called a vectorspace if it satisfies the eight axioms above. Linear algebra is an axiomatic theory which has many realizations, that is models of the eight axioms.

Theorem For any group $(G, *, e, ^{-1})$ one has that for given g and h one has that the equation $g * x = h$ has a uniques solution $x = g^{-1} * h$.

Proof: Assume that thjere is such an x such that $g * x = h.$ Then multiplication from the left by g^{-1} yields $g^{-1} * (g * x) = g^{-1} * h.$ Then by associativity we get $(g^{-1} * g) * x = e * x = x = g^{-1} * h.$This show that there is at most one such $x.$Namely $x = g^{-1} * h.$But we have that $g * (g^{-1} * h) = h$ This proves existence of such an $x.$

Theorem For vector spaces over a field F we have that

$$r.\alpha = 0 \text{ iff } r = 0 \text{ or } \alpha = 0$$

Proof. If $r = 0$ then $0.\alpha = 0.$ Here we need distributivity to connect the $0 \in F$ for addition with scalar multiplication: $0.a = (0+0).\alpha = 0.\alpha + 0.\alpha.$Now because $0.\alpha = 0.\alpha + 0$ we get by the previous theorem that $0.\alpha = 0.$

Now assume that $r \neq 0.$ Then the field element r has a multiplicative inverse $r^{-1}.$ But then $(r^{-1} \cdot r).\alpha = 1.\alpha = \alpha$ by the unitary law. If we assume that $r.\alpha = 0$ and $r \neq 0$ then $r^{-1}.r.\alpha = r^{-1}.0$ where $r^{-1}.r.\alpha = (r^{-1} \cdot r).\alpha = \alpha$ and $r^{-1}.0 = 0$ by the first part of the theorem. Thus $\alpha = 0.$

A careful analysis of the proof shows that all 8 axioms for vector spaces have been used.

Let V be any vector space over the field F and let A be any set. Assume that T is any one-one and onto map, a bijection, between V and $A.$Then the vectorspace structure of V can be transported from V to A via T :

Let α and β be elements of $A.$Then define and addition $\oplus$ on A:
$\alpha \oplus \beta = T(T^{-1}(\alpha) + T^{-1}(\beta))$

Let c be an element of F and α be an element of $A.$Then define a multiplication c_* of c and α : $c_*\alpha = T(c.T^{-1}(\alpha))$

With these definitions of addition and multiplication by field elements, the set A becomes a vectorspace like $V.$ The elements of V are given just other names taken from $A.$The operations on A are defined so that T and T^{-1} are isomorphisms:

$$T^{-1}(\alpha \oplus \beta) = T^{-1}(T(T^{-1}(\alpha) + T^{-1}(\beta)) = T^{-1} + T^{-1}(\beta), \;\; T^{-1}c_*\alpha) = T^{-1}(T(c.T^{-1}(\alpha)) \;\; = T^{-1}(\alpha))$$

Let $V = \mathbb{R}^2$ and A be the set $\mathbb{R}^2$ again but where we have forgotten that $\mathbb{R}^2$ is a vector space with the usual vector addition and multiplication by scalars. Choose a and b be arbitrarily. Then $T(x,y) = (x+a, y+b)$ is a one-one and onto map from$V = \mathbb{R}^2$ to $A = \mathbb{R}^2.$ If $\alpha = (u,v), \beta = (w,z)$ are in A then $T^{-1}(u,v) = (u-a, v-b),$ $T^{-1}(w,z) = (w-a, z-b)$ and $(u-a, v-b) + (w-a, z-b) = (u+w-2a, v+z-2b)$ and $\alpha \oplus \beta = T((u+w-2a, v+z-2b) = (u+w-a, v+z-b).$

For example $(u,v) \oplus (a,b) = (u+a-a, v+b-b) = (u,v).$ Thus with respect to $\oplus,$ the zero vector is $(a,b).$ Thus any element of $\mathbb{R}^2$ can be the zero vector.

We can map $V = \mathbb{R}^2$ to $A = (0,\infty) \times (0,\infty)$ using the exponential map:$(x,y) \to (e^x, e^y).$Then $(1,1)$ is the zero for $\oplus.$