Subspaces and Linear Systems

A subset U of a vectorspace V is called *closed* if
(a) $0 \in U$ where 0 is the zero vector of V.
(b) If α and β are elements of U then $\alpha + \beta$.
(c) If $c \in F$ and $\alpha \in U$ then $c.\alpha \in U$

Thus on U addition and multiplication of vectors can be defined, just as restrictions of those operations from V to U. That 0 belongs to U is equivalent to saying that U is non-empty: If α is any vector in U then by (c) the vector $0.\alpha = 0$ belongs to U. It is clear that U becomes a vector space in its own right. The axioms are all equations, like commutativity for addition. And these axioms hold for all vectors of V but then also for those in U.

Closed subsets of a vector space V are also called subspaces. It is easy to see the

Theorem *The intersection of closed subsets of a vector space is closed*
Proof. Let $U_i, i \in S$ be a non-empty system of closed subsets. Each U_i contains 0. Thus $0 \in U = \bigcap U_i$. If α and β are in U. Then α and β belong to evey U_i. Because each U_i is closed we have $a + \beta \in U_i$ for every i. Thus $\alpha + \beta \in U = \bigcap U_i$. Further, if $\alpha \in \bigcap U_i$ then for every U_i one has that $a \in U_i$. Because U_i is closed, $c.\alpha \in U_i$ for every i. Hence $c.\alpha \in U = \bigcap U_i$.

The intersection of the empty family of subspaces is defined as $\{0\}$ which is the smallest closed subset of V

If S is any subset of V, then there is a smallest closed subset of V which contains S. It is called the span of S and denoted as $< S >$:

$$< S >= \bigcap \{U|U \text{ closed}, S \subseteq U\}$$

If $S = (\beta_1, \ldots, \beta_k\}$ is a finite subset of V, then

$$< \beta_1, \ldots, \beta_k >= \{c_1.\beta_1 + c_2.\beta_2 + \ldots + c_k.\beta_k | c_i \in F\}$$

is the set of all *linear combinations* of the β_i. In particular, $< \alpha >= \{c.\alpha | c \in F\}$ and $< \alpha, \beta >= \{c.\alpha + d.\beta | c, d \in F\}$.
Let $V = F^n$. Closed subsets of V are given by solutions of finitely many linear homogeneous equations in n unknowns $x_1, x_2, \ldots, x_n$. Such a system looks like

$$a_{11}x_1 + a_{12}x_2 + \ldots a_{1n}x_n = 0$$
$$a_{21}x_1 + a_{22}x_2 + \ldots a_{2n}x_n = 0$$
$$\ldots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots a_{mn}x_n = 0$$

Clearly the zero vector is a solution of such a homogeneous system and if $\alpha \in F^n$ and $\beta \in F^n$

are solutions then any linear combination $c.\alpha + d.\beta$ is a solution. Linear systems are most conviniently described in matrix notation: $AX = 0$ where A is the $m \times n$ –matrix of the coefficients and X is the column of n many unknowns.

$$2x + y - 3z = 0$$

$$x - 2y + z = 0$$

are two equations in 3 unknowns which in matrix notation looks like

$$\begin{pmatrix} 2 & 1 & -3 \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We can solveany system of linear equations by

(a) multiplying any equation by some $c \neq 0$;
(b) adding a multiple of one equation to another one;
(c) interchanging the order of the equations.

Obviously these operations do not change solvability of a system (in case of inhomogeneous system) and in case of consistency these operations do not change the solution set. An inhomogeneous sytem might be unsolvable, for example the system

$$x + y = 0$$

$$2x + 2y = 1$$

is unsolvable.
The three operations (a),(b),(c) applied to the matrix A of the system $AX = 0$ lead to the elementary row operations on $A.$ In our example of two eqations in three unknowns we get

$$\begin{pmatrix} 2 & 1 & -3 \\ 1 & -2 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 1 \\ 2 & 1 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 1 \\ 0 & 5 & -5 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$

We interchanged the order of the two rows, the added $(-2) - times$ the first row to the second row , then divided the second row by 5 and finally added $2 - times$ the second row to the first row.
The new matrix stands for the equational system

$$x + 0.y - z = 0$$

$$0.x + y - z = 0$$

which is

$$x = z$$

$$y = z$$

We may choose for z anything and compute x as $z.$ Choosing for $z = 1$ we get $x = 1, y = 1, z = 1.$ The solutions of this system are given as th espan of $(1,1,1)$.
In general, let $AX = B$ be a linear inhomogeneous system . The three operations allow us to find all solutions, in case of consistency. But first we have a general
.

Theorem. Let X_1 and X_2 be solutions of $AX = B$. Then $X_1 - X_2 = X$ is a solution of the homogeneous system $AX = 0$.
Proof. $AX = A(X_1 - X_2) = AX_1 - AX_2 = B - B = 0$.

In order to solve $AX = B$ we only need to find a **particular** solution Y_0 of $AX = B$ .Then all solutions of $AX = B$ are given by $Y = Y_0 + X$ where X is any solution of $AX = 0$.

The matrix for $AX = B$ is:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{pmatrix}$$

Besides the three elementary row operations we allow also an interchange of columns of A which amounts to interchangin unknowns x_i with x_j. We must keep track of such changes. Then if $A = 0$ then in case that $B = 0$ the system is consistent and every $X \in F^n$ is a solution of $AX = 0$. In case that $A \neq 0$, we hae that there is a non-zero entry in A. We may assume that the first colunm contains a non-zero entry and we may assume that $a_{11} \neq 0$. Dividing the first row by a_{11} we achieve that the first entry in the first row is 1. Adding multiple of the first row to the other rows we get that the first column turns into the first unit vector. For example if the first entry of the second row is c_{21} then just add $-c_{21}$ of the first row to the second row, The c_{21} entry turns to 0. Similarly, for the other entries of the first column. The matrix version of our inhomogeneous system $AX = B$ is now

$$\begin{pmatrix} 1 & c_{12} & \dots & c_{1n} & c_1 \\ 0 & c_{22} & \dots & c_{2n} & c_2 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & c_{m2} & \dots & c_{mn} & c_m \end{pmatrix}.$$ We now look at the **remainder matrix**

$$\begin{pmatrix} c_{22} & \dots & c_{2n} & c_2 \\ \dots & \dots & \dots & \dots \\ c_{m2} & \dots & c_{mn} & c_m \end{pmatrix}.$$ If all c_{ij} are zero, then in case that one of the c_j for $j \geqq 2$ is different from 0 we have reached an inconsitency $0 = c_j$. We could divide the j^{th} – row by c_j to get the canonical contradiction $0 = 1$. If all $c_j = 0$ then we have reduced the system $AX = B$ only to the one equation
$x_1 + c_{12}x_2 + \ldots + c_{1n}x_n = c_1$ or $x_1 = c_1 - c_{12}x_2 - \ldots - c_{1n}x_n$. We may assign arbitray values to $x_2, \ldots, x_n$ and determine x_1 according to the equation $x_1 = c_1 - c_{12}x_2 - \ldots - c_{1n}x_n$. If we choose $x_2 = \ldots = x_n = 0$ then a particular solution of $AX = B$ is $Y_0 = (c_1, 0, \ldots, 0)$. Choosing $x_2, \ldots, x_n$ arbitraily we get $n - 1$ solutions $X_2 = x_2(-c_{12}, 0, \ldots 0), \ldots, X_n = x_n(0, 0, \ldots, -c_{1n})$ where

$$X = x_2X_2 + \ldots + x_nX_n$$

is the general solution of the homogeneous equation $AX = 0$. For example

$$2x + y + 3z + u = 5$$

is one inhomogeneous equation in 4 unknowns. It is equivalent to

$$x = \frac{5}{2} - \frac{1}{2}y - \frac{3}{2}z - \frac{1}{2}u$$

A particular solution is given by choosing $y = z = u = 0$ as $Y_0 = (\frac{5}{2}, 0, 0, 0)$. The general solution of the homogeneous eqaution is

$$X = y.(-\frac{1}{2}, 1, 0, 0) + z.(-\frac{3}{2}, 0, 1, 0) + u.(-\frac{1}{2}, 0, 0, 1)$$

and the general solution of $2x + y + 3z + u = 5$ is
$Y = Y_0 + X = (5, 0, 0, 0) + y.(-\frac{1}{2}, 1, 0, 0) + z.(-\frac{3}{2}, 0, 1, 0) + u.(-\frac{1}{2}, 0, 0, 1)$
Now continuing with the case that in
$\begin{pmatrix} c_{22} & \ldots & c_{2n} & c_2 \\ \ldots & \ldots & \ldots & \ldots \\ c_{m2} & \ldots & c_{mn} & c_m \end{pmatrix}$ not all c_{ij} are zero. Then, as before we may assume that $c_{22} \neq 0$.
Dividing the second row by c_{22} we change c_{22} entry to 1. Adding multiples of the second row to all other rows our orignal matrix equation $AX = B$ changed to

$$\begin{pmatrix} 1 & 0 & \ldots & d_{1n} & d_1 \\ 0 & 1 & \ldots & d_{2n} & d_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & d_{mn} & d_m \end{pmatrix}$$

If all $d_{ij} = 0$ for $i \gneq 3$ then all d_j for j must be zero for consistency. Otherwise we have $1 = 0$.
If $d_{ij} = 0$ for $i \gneq 3$ and $d_j = 0$ for $j \gneq 3$ then the system $AX = B$ has been solved. For example

$$x + 2z - 3u = 5$$
$$y - z - 2u = 8$$

is equivalent to

$$x = 5 - 2z + 3u$$
$$y = 8 + z + 2u$$

Hear z and u can be chosen arbitrarily. In order to get a particular solution, we may chooses $z = u = 0$ and we get

$Y_0 = (5, 8, 0.0)$ as particular solution of the inhomogeneous system

and

$X_z = z.(-2, 1, 1, 0)$ and $X_u = u.(3, 2, 0, 1)$ are two linearly independent solutions of the homogeneous system

They are linearly independent because $z.(-2, 1, 1, 0) + u.(3, 2, 0, 1) = (0, 0, 0, 0)$ forces $z = u = 0$. The general solution of

$$x + 2z - 3u = 5$$
$$y - z - 2u = 8$$

is

$$X = (5, 8, 0, 0) + z.(-2, 1, 1, 0) + u.(3, 2, 0, 1)$$

The general method of solving an equational system $AX = B$ leads in case of consistency to an equivalent system

$x_1 + c_{1(r+1)}x_{r+1} + c_{1(r+2)}x_{r+2} + \ldots + c_{1n}x_n = c_1$

$x_2 + c_{2(r+1)}x_{r+1} + c_{2(r+2)}x_{r+2} + \ldots + c_{2n}x_n = c_2$

$\ldots$

$x_r + c_{r(r+1)}x_{r+1} + c_{r(r+2)}x_{r+2} + \ldots + c_{rn}x_n = c_r$

In case of inconsistency, you have an additional equation

$0 = 1.$

The system is equivalent to

$x_1 = c_1 - c_{1(r+1)}x_{r+1} - c_{1(r+2)}x_{r+2} - \ldots - c_{1n}x_n$

$x_2 = c_2 - c_{2(r+1)}x_{r+1} - c_{2(r+2)}x_{r+2} - \ldots - c_{2n}x_n$

$\ldots$

$x_r = c_r - c_{r(r+1)}x_{r+1} - c_{r(r+2)}x_{r+2} - \ldots - c_{rn}x_n$

$x_1, x_2, \ldots, x_r$ are expressed in terms of free variables $x_{r+1}, x_{r+2}, \ldots, x_n$. If we choose them all 0, then we get a particular solution

$$Y_0 = (c_1, c_2, \ldots, c_r, 0, 0, \ldots, 0) \text{ of the inhomogeneous system}$$

The number r is called the row-rank of A. It is the *dimension* of the subspace generated by the rows of the matrix A.

The general solution of the homogeneous system is

$$X = x_{r+1}X_{r+1} + x_{r+2}X_{r+2} + \ldots + x_nX_n$$

where

$X_{r+1} = (-c_{1(r+1),} - c_{2(r+1)}, \ldots, -c_{r(r+1}, 1, 0, \ldots, 0), \ldots X_n = (-c_{1n}, -c_{2n}, \ldots, c_{rn}, 0, \ldots 1)$

are $n - r$ –many linearly independent solutions of the homogeneous system $AX = 0$. The dimension of the solution space of $AX = 0$ is $n - r$.

The general solution of $AX = B$ is

$$X = Y_0 + x_{r+1}X_{r+1} + x_{r+2}X_{r+2} + \ldots + x_nX_n$$

For the equational system $AX = B$ we use matrix A but augmented by an additional right-most column **B**: $A|B$. The three elementary row operations (a),(b),(c) transform $A|B$ into **reduced row echelon** form. SNB does this for you.

Problem 2a, page 32 of the book is an inhomogeneous system of 3 equations in 4 unknowns:

$$2x_1 - 2x_2 - 3x_3 = -2, \quad 3x_1 - 3x_2 - 2x_3 + 5x_4 = 7 \qquad x_1 - x_2 - 2x_3 - x_4 = -3 \qquad \#$$

Its matrix is

$\begin{pmatrix} 2 & -2 & -3 & 0 & -2 \\ 3 & -3 & -2 & 5 & 7 \\ 1 & -1 & -2 & -1 & -3 \end{pmatrix}$, row echelon form: $\begin{pmatrix} 1 & -1 & 0 & 3 & 5 \\ 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ according to SNB

,and now stands for the equational system:

$x_1 - x_2 + 0x_3 + 3x_4 = 5$ $\qquad$ $x_1 = 5 + x_2 - 3x_4$
$0x_1 + 0x_2 + x_3 + 2x_4 = 4$ $\qquad$ $x_3 = 4 - 2x_4$

x_1 and x_3 are expressed in term of x_2 and x_4.

A particular solution is given by choosing $x_2 = x_4 = 0$ and we get $Y_0 = (5,0,4,0)$ and the general solution of the homogeneous system has basis of $X_2 = (1,1,0,0)$, $X_4 = (-3,0,-2,1)$
The general solution of the system is

$$X = Y_0 + x_2X_2 + x_{4.}X_4 = (5,0,4,0) + x_2.(1,1,0,0) + x_4.(-3,0,-2,1)$$

The system 2b .page 32 are again 3 equations in 3 unknowns:

$$3x_1 - 7x_2 + 4x_3 = 10$$
$$x_1 - 2x_2 + x_3 = 3$$
$$2x_1 - x_2 - 2x_3 = 6$$

Its matrix is
$\begin{pmatrix} 3 & -7 & 4 & 10 \\ 1 & -2 & 1 & 3 \\ 2 & -1 & -2 & 6 \end{pmatrix}$, row echelon form: $\begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & -3 \end{pmatrix}$ which reads $x_1 = -2$; $x_2 = -4$;
$x_3 = -3$ as unique solution

problem 2(c), p. 33 is a system of 3 equations in 4 unknowns. Which is inconsistent.

$$x_1 + 2x_2 - x_3 + x_4 = 5$$
$$x_1 + 4x_2 - 3x_3 - 3x_4 = 6$$
$$2x_1 + 3x_2 - x_3 + 4x_4 = 8$$

Its matrix is
$\begin{pmatrix} 1 & 2 & -1 & 1 & 5 \\ 1 & 4 & -3 & -3 & 6 \\ 2 & 3 & -1 & 4 & 8 \end{pmatrix}$, row echelon form: $\begin{pmatrix} 1 & 0 & 1 & 5 & 0 \\ 0 & 1 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

The third row gives us the contradiction $0 = 1$. The system is not solvable.
You should do (d)-(f) of #2, p.33 using SNB.
We have seen that the solutions of a homogeneous linear system form a subspace U of dimension $n - r$ if r is the row-rank of A. And we know how to find a basis of U.
Now, given a subspace as the span $U = \langle \beta_1, \beta_2, \ldots, \beta_k \rangle$ then if $c \neq 0$ we can replace any vector β_i by $c.\beta_i$. Say $\beta_i = \beta_1$. Then $\langle \beta_1, \beta_2, \ldots, \beta_k \rangle = \langle c.\beta_1, \beta_2, \ldots \beta_k \rangle$. Indeed

$c.\beta_1 \in < \beta_1, \beta_2, \ldots, \beta_k >$ and furthermore $\beta_2, \ldots \beta_k \in < \beta_1, \beta_2, \ldots, \beta_k >$. It follws $< c.\beta_1, \beta_2, \ldots \beta_n > \subseteq < \beta_1, \beta_2, \ldots, \beta_k >$. We also have that $\beta_1 \in < c.\beta_1, \beta_2, \ldots \beta_n >$. Then as before we see that $< \beta_1, \beta_2, \ldots, \beta_k > \subseteq < c.\beta_1, \beta_2, \ldots \beta_n >$.. Thus
$< \beta_1, \beta_2, \ldots, \beta_k > = < c.\beta_1, \beta_2, \ldots \beta_n >$.
We can also add to any β_i a multiple of any other β_j. For example, replace β_1 by $\beta_1 + c\beta_2$. Then $< \beta_1, \beta_2, \ldots, \beta_k > = < \beta_1 + c\beta_2, \beta_2, \ldots, \beta_k >$. We notice that $\beta_1 + c\beta_2 \in < \beta_1, \beta_2, \ldots, \beta_k >$ but also $\beta_1 \in < \beta_1 + c\beta_2, \beta_2, \ldots, \beta_k >$ because $\beta_1 = (\beta_1 + c\beta_2) - c\beta_2 + 0.\beta_3 - \ldots 0.\beta_k$. Then as before it follows that $< \beta_1, \beta_2, \ldots, \beta_k > = < \beta_1 + c\beta_2, \beta_2, \ldots, \beta_k >$.

If $V = F^n$ then we take the $k \times n$ –matrix A which has the β_i as its rows. Because the 3 elementary row operations don't change the space U generated by the rows, the non-zero rows in the reduced row echelon form give us a basis of U. Example: Decide whether the given three vectors span $\mathbb{R}^3$:

$$A = \begin{pmatrix} -2 & 0 & 3 \\ 1 & 3 & 0 \\ 2 & 4 & -1 \end{pmatrix}, \text{ row echelon form: } \begin{pmatrix} 1 & 0 & -\frac{3}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix}$$

The 3 vectors of A span a 2 –dimensional subspace which is generated by $(1, 0, -\frac{3}{2})$ and $(0, 1, \frac{1}{2})$. We could also argue that the first vector is a linear combination of the scond and third vector. This is actually the problem 3a,page 32. We set up the problem as
$x.(1,3,0) + y.(2,4,-1) = (-2,0,3)$ which leads to 3 equations in two unknowns x and y :
$x + 2y = -2$
$3x + 4y = 0$
$-y = 3$
This is easily solvable: $y = -3, 3x = -4y, 3x = 12, x = 4;$ and $y = -3, x = 4$ solves
$4.(1,3,0) - 3.(2,4,-1) = (-2,0,3)$
Recall that a **basis** $B = \{\alpha_1, \ldots \alpha_k\}$ of a vector space V is a **linearly independent** and **generating set**.. That is
(a) $V = < \alpha_1, \ldots \alpha_k >$;
(b) $\{\alpha_1, \ldots \alpha_k\}$ is linearly independent.
Conditions (a) and (b) are equivalent to
(c) Every vector α is a **unique** linear combination of vectors in
$B : \alpha = c_1.\alpha_1 + c_2.\alpha_2 + \cdots + c_n.\alpha_n$ where the components c_i are unique.

Then one has the fundamental

Theorem. Assume that the vector space V has a finite generating set. Then V has a finite basis and all bases have the same number of elements.

There are many different proofs for this theorem. The book presents it as a Corollary of the

Replacement Theorem (Grassman). Let $G = \{\alpha_1, \ldots, \alpha_n\}$ be a generating set of $n - many$ elements and $L = \{\beta_1, \ldots, \beta_m\}$ be a linearly independent subset of V. Then
(a) $m \leq n$;
(b) There are $m - many$ vectors α in G which can be replaced by $\beta's$ and the set where $\alpha's$ are

replaced by $\beta's$ also generates V Say without loss of generality $\alpha_1, \ldots \alpha_m$ can be replaced by $\beta_1, \ldots, \beta_m$ such that $G' = \{\beta_1, \ldots, \beta_m, \alpha_{m+1}, \ldots, \alpha_n\}$ generates V.

Proof. We are going to show that one of the $\alpha's$ can be replaced by β_1. We have that $\beta_1 = c_1.\alpha_1 + c_2.\alpha_2 + \ldots + c_n\alpha_n$ because G is generating. Because $\beta_1 \neq 0$, one of the c_i must be different from zero. Without loss of generality, we may assume that $c_1 \neq 0$. But then $\alpha_1 = \beta_1 - c_2/c_1.\alpha_2 - \ldots c_n/c_1.\alpha_n$. That is $\alpha_1 \in < \beta_1, \alpha_2, \ldots, \alpha_n >$. Clearly also $\alpha_2, \ldots \alpha_n \in < \beta_1, \alpha_2, \ldots, \alpha_n >$. This shows $\{\alpha_1, \ldots, \alpha_n\} \in < \beta_1, \alpha_2, \ldots, \alpha_n >$. But then $V = < \beta_1, \alpha_2, \ldots, \alpha_n >$ and $G_1 = . \{\beta_1, \alpha_2, \ldots, \alpha_n\}$ is generating V. We now get $\beta_2 = c_1.\beta_1 + c_2.\alpha_2 + \ldots + c_n.\alpha_n$. We see that one of the coefficients for the $\alpha's$ must be different from 0 because otherwise β_2 would be a multiple of β_1, contradicting that the $\beta's$ are linearly independent. Without loss of generality we may accume that $c_2 \neq 0$. Thus $\alpha_2 \in < \beta_1, \beta_2, \ldots, \alpha_n >$. And clearly $\beta_1, \alpha_2, \alpha_3, \ldots \alpha_n \in < \beta_1, \beta_2, \ldots, \alpha_n >$. Thus $< \beta_1, \alpha_2, \alpha_3, \ldots \alpha_n >= V \subseteq < \beta_1, \beta_2, \ldots, \alpha_n >$. Hence $G_2 = \{\beta_1, \beta_2, \ldots, \alpha_n\}$ is generating etc until all $\beta's$ have replaced $\alpha's$.

If $B = \{\beta_1, \ldots \beta_k\}$ is a basis and $B' = \{\beta_1, \ldots \beta_l\}$ another one then $k \leq l$ because B is generating and B' linearly independent. But by symmetry $l \leq k$ and therefore $k = l$. All bases have the same number of elements.

Example.
Let $P_n(\mathbb{R})$ be the space of real polynomials of degree $\leq n$. It has basis $1, x, \ldots, x^n$. Thus $\dim P_n(\mathbb{R}) = n + 1$. $(1, x, \ldots, x^n$ are linearly independent because $a_0 + a_1x + a_2x^2 + \ldots + a_nx^n = 0$ where 0 is the function being constant 0. But then we would have in $a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ a polynomial with infinitely many zeroes. It follows from algebra that a polynomial of degree n has at most n –many roots.)