Linear Equations for Subspaces, Cramer's Rule

The solution set of a linear system $AX = 0$ is a linear subspace of dimension $n - r$ where r is the row-rank of A. If A is an $m \times n$-matrix then $AX = 0$ stands for m –many homogeneous equations in n –many unknowns:

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x = 0$$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = 0$$

$$\ldots$$

$$a_{n1}x_1 + a_{n2}x_2 + \ldots a_{nn}x_n = 0$$

Each equation $a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n = 0$ stands for an $n - 1$ dimensional subspace of F^n assuming that not all a_{ij} are zero. The solutions of $AX = 0$ then is the intersection of m –many hyperplanes of F^n. We also know how to find a basis of the solution space U of $AX = 0$. Without loss of generality we may assume that the first unknowns $x_1, x_2, \ldots x_r$ are expressed in terms of $x_{r+1}, x_{r+2}, \ldots, x_n$. Choosing for $x_{r+1},, x_{r+2}, \ldots, x_n$ successively 1's and zeroes we get a basis $X_{r+1,} X_{r+2}, \ldots X_n$ of U.

The easiest case is one homogeneous equation in n –unknowns. Like

$$x_1 + a_2x_2 + \ldots + a_nx_n = 0, x_1 = -a_2x_2 - \ldots - a_nx_n \qquad \#$$

and we choose successively for $x_2 = 1, x_3 = \ldots = x_n = 0$ then $x_2 = 0, x_3 = 1, \ldots x_n = 0, x_2 = x_3 = 0, \ldots, x_n = 1$ and get $n - 1$ basis vectors $X_2, X_3, \ldots X_n$ and U is the set

$$x_2X_2 + x_3X_3 + \ldots + x_nX_n$$

of all linear combinations of the X_i. For example a hyperplane of $\mathbb{R}^4$ is given by one linear equation in 4 unknowns, like

$$2x_1 + 4x_2 + 3x_3 + 6x_4 = 0 \qquad \#$$

then $x_1 = -2x_2 - 3/2x_3 - 3x_4$ and three basis vectors of the hyper plane are

$$X_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, X_3 = \begin{pmatrix} -3/2 \\ 0 \\ 1 \\ 0 \end{pmatrix}, X_4 = \begin{pmatrix} -3 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

and all linear combinations $x_2X_2 + x_3X_3 + x_4X_4$ make up the hyper plane U

Now, a subspace U of F^n may be given by m –many vectors α_1 of F^n. Thus $U = < \alpha_1, \alpha_2, \ldots \alpha_m >$. If we define an $m \times n$ matrix by

$$A = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \ldots \\ \alpha_m \end{pmatrix}$$

then elementary row operations don't change the row space, that is U. If $\dim U = r$ then there are r –many non-zero rows of the row-echelon form of A from which we can read off $n-r$ many basic solutions of $AX = 0$. This $n-r$ –dimensional subspace of F^n is the space $U^\perp = \{X | \alpha_1 \perp X, \alpha_2 \perp X, \dots, \alpha_m \perp X\}$, recall that

$$(a_1, \alpha_2, \dots, \alpha_n) \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_n \end{pmatrix} = a_1 b_1 + a_2 b_2 + \dots + a_n b_n = 0 \text{ iff } \alpha \perp \beta$$

Now let $\beta_1, \beta_2, \dots, \beta_{n-r}$ be a basis of $AX = 0$. That is of $U^\perp$. We choose these vectors as rows of a $(n-r) \times n$-matrix

$$B = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \dots \\ \beta_{n-r} \end{pmatrix}$$

Amongst the solutions of $BX = 0$ are all vectors α_i. Because $\alpha_i \cdot \beta_j = \beta_j \cdot \alpha_i = 0$. This is because the β_j are solutions of $AX = 0$. The α_i span the space U where $\dim U = r$. The solution space of $BX = 0$ has dimension $n - (n-r) = r$. Thus $BX = 0$ has solution space U.

Determinants were useful to provide explicit formulas for solutions of linear equations. Nowadays only the special case of n –equations with u –unknowns is still important. We assume that in $AX = B$ the rank of A is n. That is, for every choice of B there is a unique solution $X = A^{-1}B$. We can find X according to *Cramer's* Rule. Given is the square matrix A where A is of rank n. Then let $X = (x_1, x_2, \dots, x_n)$ be the solution for $AX = B$. We have that

$$x_1 \det(A) = \det \begin{pmatrix} a_{11}x_1 & a_{12} & \dots & a_{1n} \\ a_{2'1}x_{11} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1}x_1 & a_{n2} & \dots & a_{nn} \end{pmatrix} = \det \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n & a_{12} & \dots & a_{1n} \\ a_{2'1}x_{11} + a_{22}x_2 + \dots + a_{2n}x_n & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n & a_{n2} & \dots & a_{nn} \end{pmatrix} = \det($$

which gives us

$$x_1 = \frac{\det\begin{pmatrix} b_1 & a_{12} & \dots & a_{1n} \\ b_2 & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ b_n & a_{2n} & \dots & a_{nn} \end{pmatrix}}{\det\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{2\text{‘}1} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}}$$

and similarly for the other $x_i, i = 2, \dots n$.