June 29, 2011

Name:

1. Define that the relation R is an equivalence on the set A. Answer: R is a binary relation on A which is reflexive, symmetric and transitive.\
2. Define that π is a partition of the set A. How are equivalence relations on A and partitions of A related? Answer: a partition π of a set A is a collection of non-empty subsets of A whose union is A and any two sets of π are either the same or disjoint.
3. What is the partition of the largest equivalence $A \times A$ on A? And what is the partition for the smallest equivalence Δ on A? Answer: the largest equivalence has $\pi = \{A\}$ as partitions, the smallest equivalence $\Delta = \{(a,a)|a \in A\}$ decomposes A into singletons $\pi = \{\{a\}|a \in A\}$
4. Let $A = \{a,b,c,d,e,f,g\}$. What are the classes of the smallest equivalence relation that contains the following pairs $\{(a,c),(e,c),(d,f),(g,d),(b,e)\}$? Answer: $\{a,c,e,b\},\{d,f,g\}$
5. **a**. Let $f : A \to B$ be any function from the set A to the set B. Define a relation R_f on A by $(a,b) \in R_f$ if and only if $f(a) = f(b)$. Explain why R_f is an equivalence relation. Answer: This is obvious from basic properties of equality: $a \sim a$ because $f(a) = f(a)$, if $a \in b$, that is $f(a) = f(b)$, then $f(b) = f(a)$, that is $b \in a$.Transitivity is $a \sim b, b \sim c$ yields $a \sim c$. Because $f(a) = f(b), f(b) = f(c)$ yields $f9a) = f(c)$

   b. Let $f : \mathsf{R} \to \mathsf{R}$ be the parabola, that is $f(x) = x^2$. What do the equivalence classes look like?Answer: $\{0\},\{x,-x|x > 0\}$
6. **a**. Let $A = \{a,b,c,d,e,f,g\}$ and $B = \{1,2,\}$ and let f be the function for which one has that $f(a) = 1, f(b) = 1, f(c) = 1, f(d) = 2, f(e) = 1, f(f) = 2, f(g) = 2$. What is the partition of the equivalence relation R_f for f? Answer: $\{\{a,b,c,e\}\{d,f,g\}\}$

   b. Let $f : A \to B$ be a surjection from A to B. Assume that A has n –many elements and B has m –many elements. What can you say about the number k of equivalence classes for the equivalence relation R_f?

   i. $k = n$;

   ii. $k = m$;√

   iii. none of the above.
7. **a**. Define that P is a partial order of the set A. Answer: a partial order is a) reflexive: $a \leq a$,b) anti-symmetric: $a \leq b$ and $b \leq a$ only if $a = b$. c) transitive: $a \leq b, c \leq d$ yields $a \leq c.$,

   b. Show that the relation *a divides b* is a partial order on the set N of natural numbers. Is there a minimum or maximum of this partial order? Explain your answer. Answer We have that $a|b$ iff $\exists_k\ k \cdot a = b.$ We have $a|a$ because $1 \cdot a = a,$ if $a|b$ and $b|c$ then $a \cdot k = b, b \cdot l = c$ but then $a \cdot (k \cdot l) = c$ This gives transitivity. If $a|b$ and $b|a$ then $a \cdot k = b$ and $b \cdot l = a$ and therefore $a \cdot (k \cdot l) = a$ which gives $k \cdot l = 1$ which for natural numbers means $k = l = 1.$ Notice $1|a$ for every a and $a|0$ which makes 1 the minimum and 0 the maximum of $(N, |)$

c. Let $(P, \prec)$ be a finite partially ordered set. Explain how the partial order $\prec$ can be extended to a compatible total order $\leq$. Illustrate this process where $P = \{a,b,c,d,e,f,g\}$ and where $a \prec c, b \prec c, c \prec d, d \prec e, d \prec f, b \prec g$. Answer: We pick the set M_1 of minimal elements of P and order them arbitrarily. Then we continue with $P \backslash M_1$ etc. In the given example, $M_1 = \{a,c\}$. We set $a < b$. The set $P \backslash M_1 = \{c,d,e,f,g\}$ has $M_2 = \{c,g\}$ as minimal elements. we continue $a < b < g < c$. Then we need to find the minimal elements of $P \backslash M_1 \cup M_2 = \{d,e,f\}$ which is just $M_3 = \{d\}$. which gives us $a < b < g < c < d$. Finally $P \backslash M_1 \cup M_2 \cup M_3 = \{e,f\}$ and a total order of P is $a < b < g < c < d < e < f$.

8. **a**. What does it mean that sets a and b are equivalent? State the Cantor-Bernstein theorem. Answer: There is a bijection between the sets a and b. Cantor Bersnstein says that a and b are equivalent if there is and injection from a to b and an injection from b to a.

b. Let $\mathsf{R}^+ = \mathsf{R} \cup \{+\infty\}$ be the set of real numbers extended by a new element, called $+\infty$. Is there a bijection from R onto R^+? Explain! Answer: Yes, there is. We obviously have an injection from $\mathbb{R}$ to $\mathbb{R} \cup \{\infty\}$. In order to get an injection from $\mathbb{R} \cup \{\infty\}$ we map $\mathbb{R}$ bijectively to $[0,1)$ and $\infty$to 1. And then $[0,1]$ injectively to $\mathbb{R}$