A subspace of dimension $n-1$ in an n – dimensionmal space is called a *hyper plane*. A hyper plane of the plane $\mathbb{R}^2$ is a line, and of the three dimensional space $\mathbb{R}^3$ it is a plane.
In $\mathbb{R}^n$ the single equation

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$$

has an $n-1$ – dimensional solution space, unless all coefficients are zero. The solution space of the linear system of m –equations in n –unknowns

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ \cdots\cdots\cdots\cdots&\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

is the intersection of m –many hyper planes and this intersection is of dimension $n-r$ where r is the rank of the matrix A for this linear sytem $Ax = 0$.
Because we already know that every subspace U of $\mathbb{R}^n$ is the solution space of a linear system we have the
Theorem. Every subspace U of $\mathbb{R}^n$ is an intersection of hyper planes.

The union of subspaces U and V is in general not a subspace. The union may not be closed under sums. However the intersection is always a subspace. The union $U \cup V$ generates a subspace which is called the sum $U + V$.

$$< U \cup V >= U + V = \{u + v | u \in U, v \in V\}$$

The following theorem relates the dimensions of sum and intersectuion:

Theorem. $\dim(U+V) = \dim(U) + \dim(V) - \dim(U \cap V)$.
Proof. Let $e_1, \ldots, e_k$ be a basis of $U \cap V$. We extend it to a basis of U and to a basis of V.
So let $e_1, \ldots, e_k, u_1, \ldots, u_l$ is a basis of U; and $e_1, \ldots, e_k, v_1, \ldots, v_m$ a basis of V. We claim that

$$e_1, \ldots, e_k, u_1, \ldots, u_l, v_1, \ldots, v_m \text{ is a basis of } U + V$$

This set of $k + l + m$ –many vectors is obviously generating the sum. We need to show that it is linearly independent. So assume that

$$a_1e_1 + \cdots + a_ke_k + b_1u_1 + \ldots + b_lu_l + c_1v_1 + \ldots + c_mv_m = 0$$

In the equation

$$a_1e_1 + \cdots + a_ke_k + b_1u_1 + \ldots + b_lu_l = -c_1v_1 - \ldots - c_mv_m$$

the left-hand side is a vector in U and the right-hand side a vector in V. Thus both sides describe the same vector in the intersection $U \cap V$. Thus the left-hand side must be a linear combination of the e_i alone. This gives us $b_1 = \ldots = b_l = 0$. But then we get

$$a_1e_1 + \cdots + a_ke_k + c_1v_1 + \ldots + c_mv_m = 0$$

and all a_i and c_j must be zero because the e_i and v_j are a basis of V.
Therefore, $\dim(U+V) = k+l+m = (k+l)+(k+m)-k = \dim(U)+\dim(V)-\dim(U\cap V)$

$$\dim(U+V) = k+l+m = (k+l)+(k+m)-k = \dim(U)+\dim(V)-\dim(U\cap V)$$

We know how to find a basis of $U+V$: We create a matrix A which has the generating vectors of U and of V as rows.
Let $U = < u_1,\ldots u_r >$ and $V = < v_1,\ldots,v_r >$ then $A = (u_1|\ldots|u_r|v_1|\ldots|v_s)$. The non-zero rows of the row-echelon form of A then give us a basis of $U+V$ And we also know how to find a linear system that has $U+V$ as solution space. But how can we find a linear system for $U\cap V$?
But this is easy if we have equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{r1}x_1 + a_{r2}x_2 + \cdots + a_{rn}x_n &= 0 \end{aligned}$$

for U and equations

$$\begin{aligned} b_{11}x_1 + b_{12}x_2 + \cdots + b_{1n}x_n &= 0 \\ b_{21}x_1 + b_{22}x_2 + \cdots + b_{2n}x_n &= 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ b_{s1}x_1 + b_{s2}x_2 + \cdots + b_{sn}x_n &= 0 \end{aligned}$$

for V. The combined set of $r+s$ –many equations gives us an equational system whose solution space is $U\cap V$. And we know how to find a basis of it.

Example:
$U = < (1,1,0,-1),(0,1,3,1) >, V = < (0,-1,-2,1),(1,2,2,-2 >$
We get an equational system for U by finding a basis for the nullspace of $Ax = 0$

$A = \begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \end{pmatrix}$, nullspace basis: $\left[\begin{pmatrix} 3 \\ -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 0 \\ 1 \end{pmatrix}\right]$, Thus the two equations define U : $3x_1 - 3x_2 + x_3 + 0x_4 = 0, 2x_1 - x_2 + 0x_3 + x_4 = 0$

Similarly $B = \begin{pmatrix} 0 & -1 & -2 & 1 \\ 1 & 2 & 2 & -2 \end{pmatrix}$,has nullspace basis: $\left[\begin{pmatrix} 2 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}\right]$. Thus

the two equations define V : $2x_1 - 2x_2 + x_3 + 0x_4 = 0, 0x_1 + x_2 + 0x_3 + x_4 = 0$

All four equations together define $U \cap V$. Its matrix is $C = \begin{pmatrix} 3 & -3 & 1 & 0 \\ 2 & -1 & 0 & 1 \\ 2 & -2 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$,has

nullspace basis: $\begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$. We get that $U \cap V =< (-1,-1,0,1) >$

Recall that $U =< (1,1,0,-1),(0,1,3,1) >, V =< (0,-1,-2,1),(1,2,2,-2 >$ and $\dim(U+V)$ is the rank of

$C = \begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & -1 & -2 & 1 \\ 1 & 2 & 2 & -2 \end{pmatrix}$ which is 3. Thus

$\dim(U \cap V) = \dim(U) + \dim(V) - \dim(U+V) = 2+2-3 = 1$

which agrees with our calculations.

Example 2. Find two subspaces $U =, < u_1, u_2 >, V = (v_1, v_2 >$ of $\mathbb{R}^3$ whose intersection is the line $W =< (1,2,3) >$. Of course there are infinitely many planes that intersect in W. We want to find just two of those. If U is such a plane that contains W then it has the equation $ax_1 + bx_2 + cx_3 = 0$ where $a + 2b + 3c = 0$ because $(1,2,3)$ is on U. This gives us the equation for a, b, c :

$a + 2b + 3c = 0$, $a = -2b - 3c$ leads to two planes with normal vectors

$X_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, X_3 = \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$.

Thus the line W is the solution space (intersection) of a first plane with equation $-2x_1 + x_2 + 0x_3 = 0$ and has basis $u_1 = (\frac{1}{2}, 1, 0)$ and $u_2 = (0,0,1)$ and of a second plane $-3x_1 + 0x_2 + x_3 = 0$ and has basis $v_1 = (0,1,0)$ and $v_2 = (\frac{1}{3}, 0, 1)$. Thus

$$U =< (\frac{1}{2}, 1, 0), (0,0,1) > \text{ and } V =< (0,1,0), (\frac{1}{3}, 0, 1) >$$

It is easy to verify that $2(\frac{1}{2}, 1, 0) + 3(0,0,1) = (1,2,3)$, and $2(0,1,) + 3(\frac{1}{3}, 0, 1) = (1,2,3)$

confirming that $U \cap V = W$
Our U has equation $-2x_1 + x_2 + 0x_3 = 0$ and V has equation $-3x_1 + 0x_2 + x_3 = 0$ and the system of two equations in three unknowns

$$-2x_1 + x_2 + 0x_3 = 0$$
$$-3x_1 + 0x_2 + x_3 = 0$$

has $(1, 2, 3)$ as solution.