October 20, 2018

Test 2 Math5331

Each problem is worth 20 points. You cannot use any books, notes or calculators. You have 110 minutes to complete the test.

In the following U and V are vector spaces over the same field F.

1. Label the following statements as true or false. In each part, $T : U \to V$ is a map from the finite-dimensional vector space U to V.

(1) T is linear only if $T(0_U) = 0_V$. T

(2) T is linear if and only if $T(0_U) = 0_V$. F

(3) If T is linear then T maps a family of linearly dependent vectors to a family of linearly dependent vectors. T

(4) If $\dim(U) \geq \dim(V)$ then there is a surjective linear map from U onto V. T

(5) If $\dim(U) \leq \dim(V)$ then there is an injective linear map from U to V. T

(6) If $T : U \to V$ is linear and $\alpha_1, \ldots, \alpha_n$ a basis of U then $T(\alpha_1), \ldots, T(\alpha_n)$ is a basis of V. F

(7) If $T : U \to V$ is linear and $N(T) = U$ then $R(T) = \{0\}$. T

(8) If $T : U \to V$ is linear and $R(T) = V$ then $N(T) = \{0\}$. F

(9) If $S : U \to V$ and $T : U \to V$ are both linear then $cS + T$ is linear. T

(10) If $\dim(U) = \dim(V)$ then every linear map $T : U \to V$ is bijective. F

(11) If $\dim(U) = \dim(V)$ then there is a linear map $T : U \to V$ that maps a basis of U to a basis of V. T

(12) There is no linear map $T : \mathbb{R}^3 \to \mathbb{R}^4$ which is surjective. T

10 correct answers out of 12 gave you 20 points.

2. Let

$T : \mathbb{R}^3 \to \mathbb{R}^2$, $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Find a basis of $N(T)$ and a basis of $R(T)$?

Solution. $\dim R(T) = 2.$ because $\dim R(T) + \dim N(T) = \dim \mathbb{R}^3$ we get $\dim N(T) = 1.$ The two unit vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ form a basis of $R(T)$ and the vector

$\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = -\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is a basis of $N(T)$ because

$T\left(\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}\right) = 0$

3. Find the matrix of the linear map $T : \mathbb{R}^3 \to \mathbb{R}$, for which $T(x,y,z) = x+y+z$. Find a basis of $N(T)$ and a basis of $R(T)$.

Solution: $Mat(T) = (1,1,1)$; **a basis of** $N(T)$ **is given by a basis of** $x+y+z=0$ **which is** $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ **and** $R(T) = \mathbb{R}$

and a basis of the reals is 1.

4.Find linear maps $S,T : \mathbb{R}^2 \to \mathbb{R}^2$ such $S \circ T = T_o$ the zero transformation) but $T \circ S \neq T_0$.

Solution. **Define linear maps by matrices** $T = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, S = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, that is

$T(e_1) = e_2, T(e_2) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, S(e_1) = e_1,$

$S(e_2) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. **Then** $S \circ T(e_1) = S(e_2) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, S \circ T(e_2) = S\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$

$T \circ S(e_1) = T(e_1) = e_2, T \circ S(e_2) = T\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ **and we also have that**

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

5. Find the general solution of the linear system:

$$x+y+z = 1$$
$$x-y-z = 1$$

Solution. $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$ **is the matrix of this linear sytem**. **It has the row echelon form**: $\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$, **which**

gives the equations $x = 1+0z, y = 0-1z$. **All solutions are given by**

$X = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + z\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$ **which is a line**.

6. Find a basis of $x_1 + x_2 + x_3 + x_4 + x_5 = 0$

Solution The matrix of the equation is $\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \end{pmatrix}$ **which stands for** $x_1 = -x_2 - x_3 - x_4 - x_5$ **and gives us a basis of four vectors**

$$\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

7. Let $T : U \to V, S : V \to U$ be linear and $S \circ T = id_U$. Prove that $N(T) = \{0\}$ and $R(S) = U$

Solution: **Let** $\alpha \in U$. **Assume that** $T(\alpha) = 0$. **then** $S \circ T(\alpha) = 0$ **But** $S \circ T = id_U$, **therefore** $\alpha = 0$ **That is** $N(T) = 0$. **Let** $\alpha \in U$. **Because of** $S \circ T(\alpha) = \alpha$ **we have that** $S(\beta) = \alpha$ **for** $\beta = T(\alpha)$. **That is** $R(S) = U$, **e.g.**, S **is surjective**.

8. Let $T : \mathbb{R}^4 \to \mathbb{R}$ be linear. Show that there exists numbers a, b, c, d such that

$$T(x,y,z,u) = ax + by + cz + du$$

Solution: **The matrix for** T **is a** $1x4$ –**matrix**. $A = (a,b,c,d)$ **where** $a = T(1,0,0), b = T(0,1,0), c = T(0,0,1), d = T(0,0,0,1)$ **and**

$$T(x,y,z,u) = (a,b,c,d)\begin{pmatrix} x \\ y \\ z \\ u \end{pmatrix} = ax + by + cz + du.$$

9. Let $T : U \to U$ be linear and assume that $N(T) = R(T)$ Prove that $\mathbf{dim}(\mathbf{U})$ must be even.

Solution: **According to the dimension equality we have that** $\dim N(T) + \dim R(T) = \dim U$. **If** $N(T) = R(T)$ **then** $\dim N(T) = \dim R(T)$ **thus** $2\dim(N(T) = 2\dim(R(T) = \dim(U)$. **Thus** $\dim(U)$ **must be even**.

10. Assume that $N(T) = R(T)$. Prove that $T^2 = 0$. **Solution**: **Assume** $N(T) = R(T)$. **Let** $T(T(\alpha)) = \beta$. **Because** $T(\alpha) \in R(T) = N(T)$ **we get** $\beta = 0$. **Thus** $T^2 = 0$.