March 24, 2018

Name:

Test 2, Math 5330

This test is worth **200** points. You have **90** minutes to complete the test. You cannot use any books or notes.

1. Which of the direct products are cyclic? Explain your answers.

a. $Z_2 \times Z_3$. Answer: Yes, orders are relatvely prime.

b. $Z \times Z$. Answer; No, There is no generator. Only possible ones, like $(-1,1)$, don't generate elements where only one comonent is $0'$

c. $Z_2 \times Z_4$. Answer: No, orders are not relatively prime.

2. Calculate the order of $(4,9)$ in $Z_{16} \times Z_{17}$. Answer: $68 = 4 \cdot 17; o(4) = 16/4, o(9) = 17/((9,17) = 17/1 = 17$

3. Let $a,b,c,d,e,f,g\ h$ be pairwise different What is the order of $(a,b,c)(d,e)(f,g,h)$? Answer: $6 = \mathrm{lcm}(3,2)$

4. Is $Z \times \mathbf{S}_3$ commutative? Explain your answer! Answer: No, S_3 is not commutative. See next problem.

5. Let G and H be groups. Assume that $G \times H$ is commutative. Must it be the case that G and H are commutative? You must explain your answer. Answer: Yes, $(g,h)(g',h') = (g',h')(g,h)$iff $(gg',hh') = (g'g,h'h)$ iff $gg' = g'g$ and $hh\prime = h\prime h$

6. Let $f : A \longrightarrow B$ is injective. Can you find a function $g : B \to A$ such that $g \circ f : A \longrightarrow A$ is the identity on A? Explain your answer. Answer: Yes,. Define $g : B \to A, g(b) = a$ if $f(a) = b$ and $g(b) = a_o$ where $a_0 \in A$ arbitrarily chosen. Then $g(f(a)) \;\; = g(b) = a$ for any $a \in A$. Elements b$\in$ B which are not in the inage of f don't play a role.

7. Let $\mathbf{G} = \mathbf{Z}_3 \times \mathbf{Z}_2$. We take addition $\oplus$ as operation.

a. List all elements of **G**. Answer: $(0,0),(0,1),(1,0),(1,1),(2,0),(2,1)$

b. Show that $\mathbf{H} = \{(0,0),(0,1)\}$ is a subgroup of **G**. Answer: $(0,1) \oplus (0,1) = (0,0)$. This also shows $-(0,1) = (0,1)$

c. List the right cosets of **H** in **G**.Answer:mThefre are 3 cosets, $H = \{(0,0),(0,1), H + (1,0) = \{(1,0),(1,1)\}, H + (2,0) = (2,0),(2,1)\}$

8. Let **G** be any group. Let $f : g \mapsto g^{-1}$, be the map which sends an element $g \in \mathbf{G}$ to its inverse. Is this map bijective? Prove your answer. Answer: Yes. It is injective: $g^{-1} = g_1^{-1}$ implies $g = g_1$ take the inverse on both sides. Given any $z \in G$ then $f(z^{-1}) += (z^{-1})^{-1} = z$ shows surjectivity.

9. **a**. Define that R is an equivalence relation on the set A. Answer: R is reflexive, aRa, R is symmetric $aRb \to bRa$, R is transitive $aRb \wedge bRc \to aRc$

b. Let $f : A \to B$ be any map from the set A to the set B. Prove that a_1Ra_2 iff $f(a_1) = f(a_2)$ defines an equivalence relation on A. Answer: This is utterly trivial: $f(a) = f(a), f(a_1) = f(a_2) \to f(a_2) = f(a_1), f(a_1) = f(a_2) \wedge f(a_2) = f(a_3) \to f(a_1) = f(a$

c. Let f be $\sin\ : \mathbb{R} \to \mathbb{R}$. What is the equivalence class of 0?Answer: $[0] = \{k\pi | k \in \mathbb{Z}\}$.

10. **a**. Find the order of the following permutation in S_{10}:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & 7 & 5 & 6 & 1 & 4 & 9 & 10 & 8 & 2 \end{pmatrix}$$

Answer: $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & 7 & 5 & 6 & 1 & 4 & 9 & 10 & 8 & 2 \end{pmatrix} = (1,3,5)(2,7,9,8,10)(4,6)$

$o(\pi) = \mathrm{lcm}(3,5,2) = 30$

b. Is this permutation even or odd?Answer: It is odd. It is the product of two even and one odd permutation.