The Homomorphism Theorem for Groups

Given a map $f : A \to B$ between sets A and B then it defines an equivalence relation on $A : a_1 \sim a_2$ iff $f(a_1) = f(a_2)$. It is called $\ker(f)$. The set A is partioned into classes on which f is constant.

For example, let T be the temperature function on the US. Then the equivalence classes are the isotherms, that is, locations on which the temperature is the same.

A more geometric example is given by the distance function $d : \mathbb{R}^2 \to \mathbb{R}^{\geq 0}, (x,y) \mapsto \sqrt{x^2+y^2}$. The classes for $\ker(d)$ are points of the plane where d is constant. That is they lie on the circle with radius $r = d(x,y)$.

For a given map $f : A \to B$ the set of equivalence classes gives us the 'factor set" $A/\ker(f) = \{[a] | a \in A\}$ where $[a] = \{a' | f(a) = f(a')\}$

The map $\phi_f : A \twoheadrightarrow A/\ker(f), a \mapsto [a]$ is called the canonical projection for f. We have an injection $\varepsilon_f : A/\ker(f) \longrightarrow B, [a] \longrightarrow f(a)$ Notice that ε_f is well defined. On $[a]$ the map f is constant.

We have

$$f = \varepsilon_f \circ \phi_f$$

That is, any map decomposes into a surjection follows by an injection. This is the "general homomorphism theorem".

For example, we have that $d(x,y)$ is the length of the radius of the circle on which (x,y) is located.

For a homomorphism $f : G \longrightarrow H$ between groups, we have that $\ker(f)$ is the equivalence relation which says that $a_1 \sim a_2$ iff $f(a_1) = f(a_2)$. But then $f(a_1)(f(a_2))^{-1} = f(a_1 a_2^{-1}) = e_B$. That is $a_1 a_2^{-1} \in N_f = \{a | f(a) = e_B\}$. We have that N_f is the subset of the group containing all elements which are mapped under f to the unit e_B of the group B. In particular, $e_A \in N_f$. Actually, N_f is the equivalence class of e_A for the equivalence relation $\ker(f)$. N_f is called the kernel $Ker(f)$. Notice that $Ker(f)$ is a subset of A while $\ker(f)$ is a binary relation. N_f is a normal subgroup and it defines an equivalence relation, $\ker(f)$, whose equivalence classes are the cosets $aN = Na$ on which f is constant $f(a)$. The set of equivalence classes is a group where $a_1N \bullet a_2N = a_1a_2N$. The map $\phi_f : A \longrightarrow A/\ker(f), a \mapsto [a] = aN$, is a surjective homomorphism and the map $\varepsilon_f : A/N \rightarrowtail B, aN \mapsto f(a)$ is an injective homomorphism. We need to show that ε_f is a homomorphism. Injectivity is obvious. Now $\varepsilon_f(a_1N \bullet a_2N) = \varepsilon_f(a_1a_2N) = f(a_1a_2) = f(a_1)f(a_2) = \varepsilon_f(a_1N)\varepsilon_f(a_2N)$.

If $f : A \twoheadrightarrow B$ is a surjective homomorphism then ε_f is also surjective. Let $b \in B$. Then $b = f(a)$ for some $a \in A$ and $\phi_f(a) = aN$ where $\varepsilon_f(aN) = f(a) = b$. From the homomorphism theorem we get that A/N is isomorphic under ε_f. In short:

If $f : f : A \twoheadrightarrow B$ is surjective then $A/N_f \cong B$

This is usually called the "Fundamental Theorem on Group Homomorphisms".

For example, let $C =< x >$ be a cyclic group which is generated by x. That is, $C = \{x^n | n \in \mathbb{Z}\}$. Then $e : \mathbb{Z} \to C, n \mapsto x^n$ is a surjective homomorphism between $(\mathbb{Z},+)$

and C. We have $Ker(e) = N$ where N is a subgroup of $\mathbb{Z}$. That is $N = k\mathbb{Z}$ where k is the smallest non-negative number in $\{n|x^n = 1\}$. That is $C = \{x^0 = 1, x, \ldots, x^{k-1}\}$ in case that C has k –many elements. That is C is like $\mathbb{Z}_k$ and we get

$$\mathbb{Z}/k\mathbb{Z} \cong \mathbb{Z}_k$$