Induction and Recursion

The successor of a set A is a set with one new element added. But what should it be? We assume as an axiom that a set A contains elements of *lower rank* than A. We are not going into the definition of rank. This is part of a course in axiomatic set theory. So A is not an element of $\dot{A}$ Thus $A \cup \{A\}$ contains one more element than A. We define

$$A^+ = A \cup \{A\}$$

as the successor of the set A.
Examples: (a) $\emptyset^+ = \emptyset \cup \{\emptyset\} = \{\emptyset\} = 1$ (b) $1^+ = \{\emptyset\} \cup \{\{\emptyset\}\} = \{\emptyset, \{\emptyset\}\} = \{0,1\} = 2$
(c) $2^+ = \{0,1\} \cup \{\{0,1\}\} = \{0,1,2\} = 3$
$n^+ = \{0,1,\dots,n-1\}^+ = n+1$

A set is I called *inductive* if $\emptyset \in I$ and if $A \in I$ then $A^+ \in I$. An inductive set must contain $0,1,2,\dots n, n+1,\dots$ all natural numbers. thus an inductive set must be infinite.
The axioms of set theory stipulate that there is an inductive set. This is called the axiom of infinity. But then there is a smallest inductive set, namely the intersection ω of all inductive sets.

$$\omega = \bigcap\{I | I \text{ inductive}\}$$

Clearly, all natural numbers $1,2,3,\dots$ are members of ω. We assume that ω consists only of natural numbers n. This gives us the

Principle of Mathematical Induction. Assume that A is a set of natural numbers. Then $A = \omega$ = set of all natural numbers if the following are true:
Basis step: $0 \in A$
Inductive step: If $k \in A$ then $k^+ = k+1 \in A$

We assume $A \subseteq \omega$ and get $A = \omega$ because A is inductive and ω is the smallest inductive set.
Any natural number $n > 0$ contains n –many predecessors: $0,1,\dots,n-1$. From this we get the

Well-ordering Principle: *Any non-empty set of natural numbers contains a smallest element.*

A function $f : \omega \to \omega$ is *recursively* (or by induction) defined if $f(0)$ is given and its value at n can be obtained from its values at smaller numbers.
Example: Addition of natural numbers $n+m$ can be defined by

$$n + 0 = n$$

$$n + m^+ = (n+m)^+$$

In particular, $n + 1 = (n+0)^+ = n^+$;
In order to see that functions can be defined inductively, one has that partial functions $f_n : \{0,1,\dots,n-1\} \to \omega$ are defined and $f = \cup f_n$. One needs to show that if $k < n, k < m$ then $f_n(k) = f_m(k)$.

Well formed formulae can be defined recursively as follows

a) All propositional variables $p_1, p_2, \ldots$ are formulas

b) If α β are formulas then $\neg\alpha, (\alpha \wedge \beta), (\alpha \vee \beta), (\alpha \rightarrow \beta), (\alpha \Leftrightarrow \beta)$ are formulas

c) All formulas are obtained by finite many applicatio