Practice Test1 and Transport Principle

Let A be an arbitrary set and $(B,*)$ be a group. let $f : A \to B$ be any bijection, that is one-one and onto map from A to $B.$ We then can transport, better pull-back, the group structure of B to $A.$ This is done in a most natural way. If a_1 and a_2 are any two elements of A we define their product $a_1 * a_2$ by multiplying their images in $B,$ that is we take $f(a_1) * f(a_2)$ which is a product in B and then take the inverse image of the product. That is:

$$a_1 * a_2 = f^{-1}(f(a_1) * f(a_2))$$

For example, $A = \{a,b\}$ can be made to a group Using the two element group $\mathbb{Z}_2 = \{0,1\}$ with addition where $1+1 = 0.$ We take the bijection between A and $\mathbb{Z}_2$ which maps a to 0 and b to $1.$ Then $b * b = f^{-1}(f(b) + f(b)) \ = f^{-1}(1+1) = f^{-1}(0) = b.$ similary for the other products. Here A becomes a tw-element group where a serves as 0 and b as unit.

A more sophisticated example was given as an exercise. The question is: Is $\mathbb{R}\backslash\{1\}$ with the operation $a * b = a + b - ab$ a group. We first need to show that $a + b - ab$ is never 1 in case that a and b are different from $1.$ Indeed, $a + b - ab = 1$ leads to $a + b(1-a) = 1,$

$b(1-a) = 1 - a.$ If $a \neq 1$ then $b = 1.$ Now we know that $\mathbb{R}\backslash\{0\}$ is a group under multiplication and $f : \mathbb{R}\backslash\{1\} \to \mathbb{R}\backslash\{0\}$ is the straight line $f(x) = 1 - x.$ and $f^{-1}(y) = 1 - y.$ Then

$$a * b = (1 - (1-a)(1-b) = 1 - (1 - a - b + ab) = a + b - ab)$$

This makes $\mathbb{R}\backslash\{1\}$ to a group which looks like the reals different from zero together with multiplication. There is no need to show associativity and and the other group axioms. The unit of this group is the conter image of $y = 1$ so it is 0 in $\mathbb{R}\backslash\{1\}.$

A bijection $\mathbb{Z} \longrightarrow \mathbb{Z}$ is the map $x \mapsto x - 1$ with inverse $y \longmapsto y + 1.$ An new addition $*$ then is given bythe formula

$$a * b = (a-1) + (b-1) + 1 = a + b - 1$$

The new addition on $\mathbb{Z}$ has the same properties as $(\mathbb{Z},+).$ What is the cyclic generator of $(\mathbb{Z},*)$?

Practice test 1:

1. State the Well-Ordering Principle for Z^{+}. Answer: For the test, study first and second form of induction.
2. Prove by induction that $1 + 3 + 5 + \cdots + (2n+1) = (n+1)^2$. **Answer**: This is obvious for $n = 1 : 1 + 3 = 4 = 2^2.$ Assume the claim for n Then: $1 + 3 + 5 + \cdots + (2n+1) + (2(n+1)+1) = (n+1)^2 + 2n + 3 = n^2 + 2n + 1 + 2n + 3 = n^2$
3. Prove by Mathematical Induction that if a set S has n elements then S has 2^n –many subsets. Answer: If S has no element, then there is only one subset, namely the empty set. Assume the claim for any set S with n –elements. Add any element c to the set $S.$ Then the subsets of $S \cup \{c\}$ are the subsets A of S plus all sets $A \cup \{c\}.$ Thus $S \cup \{c\}$ has twice as many subsets as S has. S has by induction hypothesis 2^n many subsets thus

$S \cup \{c\}$ has $2 \times 2^n = 2^{n+1}$ –many subsets.

4. Is division a commutative operation on the set R^+ of positive numbers? Is it associative? Explain your answers. Answer: of course, division is not commutative, $1/2 \neq 2/1$. It is not associative, $((1/2)/3) \neq 1/(2/3)$

5. Define that $(G, *)$ is a group. You can also state that $(G, *, ^{-1}, e)$ is a group. Answer: RTead this in the book.

6. Define the additive group $(\mathrm{Z}_n, \oplus)$ of integers modulo n. You have to state exactly what its elements are, how $\oplus$ is defined, what the identity is, and how the inverse of an element is defined. **Answer**: Read this in the book.

7. **a**. Find the additive inverse of 40 modulo 48. **Answer**: $-40 \bmod 48 = 8 \bmod 48$

b. Solve $15 + x = 7$ modulo 48. Answer: $x = 7 - 15 = -8 \bmod 48 = -8 + 48 = 40 \bmod 48$

8. Let $(G, *)$ be a group such that $x^2 = e$ for all $x \in G$. Show that G is abelian. Answer: $x^2 = e$ is the same as $x = x^{-1}$ for every x. Thus $xy = (xy)^{-1} = y^{-1}x^{-1} = yx$

9. Let G be a finite group, and consider the multiplication table for multiplication of G. Prove that every element of G occurs precisely once in each row and once in each column. **Answer**: This is just that given some x and z you can find some y such that $xy = z$. This is clear, $y = x^{-1}z$. This takes care for rows. For columns you argue that given any y and z you can finds some x such that $xy = z$. Here $x = y^{-1}z$.

10. Find a multiplication table on the set $A = \{a, b, c, d\}$ where every element of A occurs precisely once in each row and once in each column but where A is **not** a group. You have to explain why your algebra $A = (\{a, b, c, d\}, *)$ is not a group. **Answer**: Optional puzzle. Not needed for the test.