Truth Tables, Propositional Fuctions

A *Proposition* is a sentence which is either true or false. In order to be true or false, the sentence has to be a declaration, like

Houston is the largest city in Texas
Houston is the capital of Texas

While the first sentence is true, the second one is not. A mathematical equation may be a proposition if it contains no variable, like

$1 + 2 = 3$

while

$1 + x = 3$

is not a proposition. The truth of a proposition may be unknown, like **There are infinitely many prime twins**, like $5, 7;\ 11, 13; 17; 19, \ldots.$ or **Trump will be re-elected in 2020**.
Using logical operators, new propositions can be formed from existing ones. They are then called **compound propositions**. Example for logical operators are:
Negation,$\neg, \neg p$ is read as: It is not the case that p
Conjunction, $p \wedge q,$ is read as: p and q
Disjunction, $p \vee q,$is read as: p or q
Implication, $p \to q,$is read as: If p then q

Negation is a unary operaror and has the following truth table:

p	$\neg p$
T	F
F	T

$\neg p$ is true only if p is false.
For $p =$ Houston is the capital of Texas the truth value for p is F so $\neg p =$ **It is not the case that Houston is the capital of Texas**, is a true statement.of course we just can say that **Houston is not the capital of Texas**.

The truth table for the conjunction is

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Both propostions, p as well as $q,$ have to be true in order for the proposition $p \wedge q$ to be true.

The truth table for the disjunction is

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Both propostions, p as well as $q,$ have to be false in order for the proposition $p \vee q$ to be false.

The truth table for implication is

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

The *implication* $p \to q$ is false if p is true but q is false. Otherwise $p \to q$ is true. The book lists about a dozen versions for expressing the implication $p \to q.$ The most often used phrases for $p \to q$ are:

If $p,$ then q
p is sufficient for q
q is necessary for p
p implies q
q in case that p
p only if q
q if p

The biconditional $p \leftrightarrow q$ has the truth table

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

That is, $p \leftrightarrow q$ is true only if p and q have the same truth values, false otherwise. "p if and only if q" are the two propositions: "p if q", which is $q \to p$ and "p only if q, which is $p \to q$.

A compound proposition in propositional variabes $p_1, p_2, \ldots, p_n$ is called a **tautology** if for any choice of the truth values for the p_i the truth value for p is T.. The proposition $p \vee \neg p$ is a tautology: Clearly , $p \vee \neg p$ is T in case that p has truth value T and if p is F then $\neg p$ is T and therefore $p \vee \neg p$ is T.
A compound proposition in propositional variabes $p_1, p_2, \ldots, p_n$ is called a **contradiction** if for any choice of the truth values for the p_i the truth value for p is F. The proposition $p \wedge \neg p$ is a **contradiction**.

Two propositions p and q in propositional variables $p_1, \ldots, p_n$ are **logically equivalent** if for every choice of truth values for the p_i the truth values for p and q are the same. That is the same as saying that $p \leftrightarrow q$ is a tautology. For this we write $p \equiv q$.

De Morgan's laws for propositions are

$$\neg(p \wedge q) \equiv \neg p \vee \neg q, \neg(p \vee q) \equiv \neg p \wedge \neg q$$

This is easily seen using truth tables:

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$\neg p \vee \neg q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

For sets we have equations, like $\overline{A \cap B} = \overline{A} \cup \overline{B}$ which says the complement of the intersection of two sets is the union of the complements: If an element is not in the intersection then it doesn't belong to one of the sets. Which is quite obvious.

There are 16 different propositional functions in two variables p and q. It is quite remarkable tat they are all obtainable using just three, namely negation, conjunction and disjunction.

Theorem. Any propositional function $f(p_1, p_2, \ldots, p_n)$ is equivalent to a disjunction of a conjunction in p_i and $\neg p_i$.
Proof. If $f(p_1, p_2, \ldots, p_n) = T$ for a certain choice of truth values $v(p_i)$ for p_i which are either T or F then choose p_i if $v(p_i) = T$ and $\neg p_i$ if $v(p_i) = F$. Then take the conjunction of the p_i or $\neg p_i$ and the the disjunction of thos terms where $f(p_1, p_2, \ldots, p_n) = T$.
For example, if $n = 3$ and $f(p_1, p_2, p_3)$ has the truth table

p_1	p_2	p_3	$f(p_1, p_2, p_3)$
T	T	T	F
T	T	F	F
T	F	T	T
F	T	T	T
T	F	F	T
F	T	F	F
F	F	T	F
F	F	F	F

Then

$f(p_1, p_2, p_3) \equiv (p_1 \wedge \neg p_2 \wedge p_3) \vee (\neg p_1 \wedge p_2 \wedge p_3) \vee (p_1 \wedge \neg p_2 \wedge \neg p_3)$ is the **disjunctive normal form** for $f(p_1, p_2, p_3)$.
We also have that
$\neg f(p_1, p_2, p_3) \equiv (p_1 \wedge p_2 \wedge p_3) \vee (p_1 \wedge p_2 \wedge \neg p_3) \vee (\neg p_1 \wedge p_2 \wedge \neg p_3) \vee (\neg p_1 \wedge \neg p_2 \wedge p_3) \vee ($
and therefore by De Morgan:
$f(p_1, p_2, p_3) \equiv (\neg p_1 \vee \neg p_2 \vee \neg p_3) \wedge (\neg p_1 \vee \neg p_2 \vee p_3) \wedge (p_1 \vee \neg p_2 \vee p_3) \wedge (p_1 \vee p_2 \vee \neg p_3) \wedge$
is the **conjunctive normal form** for $f(p_1, p_2, p_3)$
We have that $p \to q$ is not true in case that p is T and q is F. That is $\neg(p \to q) \equiv p \wedge \neg q$ therefore

$$p \to q \equiv \neg p \vee q$$

Of course this can be also verified using truth tables.

We say that negation, conjunction and disjunction are **functionally complete** because any propositional function is equivalent to one that contains only these logical operators.

□