4. Sets and Functions

We are dealing here with what is called *naive* set theory. Here a set is any unordered collection of objects. If a is an element of the set A we write $a \in A.$ Besides equality $=$ the membership relation $\in$ are the only relatioms used in set theory. We assume

Leibnitz Extensionality Axiom:

$$A = B \text{ iff } \forall x(x \in A \leftrightarrow x \in B)$$

That is, two sets are equal if and only if they contain the same elements. Examples for sets are

$\mathbb{N} = \{0,1,2,\dots\},$ the set of **natural numbers**

$\mathbb{Z} = \{\dots,-2,-1,0,1,2,\dots\}$ the set of **integers**

$\mathbb{Q} = \{\frac{p}{q}|p \in \mathbb{Z}, q \in \mathbb{Z}, q \neq 0\}$ the set of **rational numbers**

$\mathbb{R}$ the set of **real numbers**

$\mathbb{C}$ the set of **complex numbers**

The order of elements in a set does not matter. For example $\{1,2,3\}$ and $\{2,3,1\}$ denote the same set of the first three positive natural numbers.

The elements of a set may be themselves sets. $\{\mathbb{N},\mathbb{Z}\}$ is a set with two elements, each of which is a set.

The empty set is the uniques set which does not have any element:

$$\emptyset = \forall x(x \notin \emptyset)$$

By extensionality, there is only one empty set.

We can define all natural numbers by starting with

$0 = \emptyset,$ then $1 = \{\emptyset\}, 2 = \{\emptyset,\{\emptyset\}\} = \{0,1\}, 3 = \{0,1,2\} = \{\emptyset,\{\emptyset\},\{\emptyset,\{\emptyset\}\}$ in general, $n^+ = \{0,1,\dots,n\}$

In mathematics, every object should be a set. We just saw that this is true for natural numbers.

The set A is a **subset** of the set B if every element of A is an element of B for which we write $A \subseteq B$:

$$A \subseteq B \text{ iff } \forall x(x \in A \to x \in B)$$

We have $\emptyset \subseteq A$ for every set $A.$ In particular

$$\emptyset \subseteq \emptyset$$

while $\emptyset \in \emptyset$ is false. The empty set does not contain any element. We also note that $A \subseteq A$ for every set $A.$

For every set $A,$ the power set $P(A)$ of A is the set of all subsets of A :

$$P(A) = \{S|S \subseteq A\}$$

For every set $A, \emptyset \in P(A)$ and $A \in P(A).$ The empty set is the smallest subset of A while A is the largest subset of $A.$

Theorem. If A is a finite set of n –many elements then $P(A)$ has 2^n –many elements.
This is easy to see. If we add to A a new element, say ω, then the subsets of $A \cup \{\omega\}$ are the subsets S of A and the sets $S \cup \{\omega\}$ where ω has been added to subsets of A. That is adding a new element to A, doubles the number of subsets of A.

Given any element a the set $\{a\}$ which consists only of a is called a **singleton**. Given a and b then the set $\{a,b\} = \{b,a\}$ is called a doubleton. Note that the order of a and b in $\{a,b\}$ are irrelevant. What matters is that the only elements are a and b.
If we wish to list a and b in a specific order, say a first and then b, then we talk about the ordered pair (a,b). Formally,

$$(a,b) = \{\{a\},\{a,b\}\}$$

This is Kuratowski's definition of the ordered pair (a,b). The first component a of (a,b) is the only element of the singleton $\{a\}$ while the second component is the element in the doubleton $\{a,b\}$.

Given sets A and B then the Cartesian product of A and B is the set $A \times B$ of all ordered pairs (a,b) where $a \in A$ and $b \in B$

$$A \times B = \{(a,b) | a \in A, b \in B\}$$

The number of elements of a finite set A is called the **cardinality** of A and Is denoted by $|A|$.
We have $|\emptyset| = |0| = 0, |n| = n, |A \times B| = |A| \cdot |B|$.

Given sets A and B, the **union** $A \cup B$ of A and B is the set of all elements which belong to A or B (or both):

$$A \cup B = \{x | x \in A \vee x \in B\}$$

The **intersection** $A \cap B$ is the set of all elements which belong to both A and B :

$$A \cap B = \{x | x \in A \wedge x \in B\}$$

Assume that our sets X are subsets of a universal set $U : X \subseteq U$ Then the **complement** $\overline{A}$ is the set of all elements of U that don't belong to A.
Union and Intersection can be generalized to more than two sets. Let S be any system of sets. Then:

$$\cup S = \{x | x \in A \text{ for some } A \in S\}, \cap S = \{x | x \in A \text{ for all } A \in S$$

We notice DeMorgan's laws for sets:

$$\overline{A \cup B} = \overline{A} \cap \overline{B} \quad \text{and} \quad \overline{A \cap B} = \overline{A} \cup \overline{B},$$

Given sets A and B a function f from A to B assigns to every element $a \in A$ a unique element $b \in B$ as image $f(a)$. Notation:

$$f : A \to B, a \mapsto f(a) \in B$$

If $f : A \to B, g : B \to C$ then we can form the **composition** $g \circ f : A \to C, a \mapsto g(f(a))$.

A function $f : A \to B$ is called one-one or **injective**, if $f(a_1) = f(a_2)$ only if $a_1 = a_2$.
A function $f : A \to B$ is called onto or **surjective** if for every $b \in B$ there is some $a \in A$ such that $f(a) = b$.
A function $f : A \to B$ is called **bijective** if it is injective and surjective. Then for every $b \in B$ there is a unique $a \in A$ such that $f(a) = b$. For a bijective function we have an inverse $f^{-1} : B \to A$, such that $f^{-1}(f(a)) = a, f(f^{-1}(b)) = b$.

For every function $f : A \to B$ we have a subset of $A \times B$ which is called the graph of f.

$$graph(f) = \{(a, f(a)) | a \in A\}$$

Assume that for $f : A \to B$ one has a function $g : B \to A$ such that $g \circ f = id_A$ where $id_A = A \to A, a \mapsto a$ is the identity on A. Then f is injective and g is surjective. Assume $f(a_1) = f(a_2)$. Then $g(f(a_1)) = g(f(a_2))$, therefore $a_1 = a_2$. That is f is injective. Now let $a \in A$. Then $g(f(a)) = a$ shows $g(b) = a$ for $b = f(a)$. That is, g is surjective.

A subset R of $A \times B$ is the graph of some function $f : A \to B$ if whenever $(a, b) \in R$ and $(a, c) \in R$ then $b = c$.