Let V and W be vector spaces over a field F A map $T : V \to W$ is called linear if

$$T(a.\alpha + b.\beta) = a.T(\alpha) + b.T(\beta)$$

We can add linear maps and multiply by scalars:

$$T + S : V \to W, \alpha \mapsto T(\alpha) + S(\alpha), a.T : V \to W, \alpha \mapsto a.T(\alpha)$$

With these operations th linear maps from V to W form a vector space, denoted as $L(V,W)$. If V has dimension n and W has dimension m then $\dim L(V,W) = m \times n$.
Linear maps can be identified with matrices. Let $\gamma = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ be a basis of V and $\sigma = \{\beta_1, \beta_2, \cdots, \beta_m\}$ be a basis of W then let

$$T(\alpha_j) = \sum_{i=1}^{m} A_{ij}\beta_i$$

A is a matrix with m rows and n columns whose entries are the A_{ij}.
A linear map is uniquely determined by it images on the basis:

If $\alpha = x_1\alpha_1 + x_2\alpha_2 + \cdots + x_n\alpha_n$ then $T(\alpha) = x_1T(\alpha_1) + x_2T(\alpha_2) + \cdots + x_nT(\alpha_n)$

If $A = Mat(T)$ then

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1j} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2j} & \cdots & A_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{m1} & A_{m2} & \cdots & A_{mj} & \cdots & A_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ \cdots \\ x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \cdots \\ y_m \end{pmatrix}$$

where the y_j are the components of $T(\alpha)$ with respect to the basis γ.
Central for linear maps is the

Theorem. Two linear maps that agree on a basis are equal. Given a basis $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ of V and n –vectors $\{\beta_1, \beta_2, \cdots, \beta_n\}$ in W then there is one and only one linear map $V \to W$ which extends $\alpha_i \mapsto \beta_1$.

Linear maps can be composed

$$T : V \to W, S : W \to R, S \circ T : U \to R, \alpha \mapsto S(T(\alpha)$$

The composition of linear maps is linear. The matrix of the composition is the product of the matrices.
Give two vector vector spaces of the same dimensions, there is a one-one and onto (bijective) linear map between them. For example, the vectors space $P_n(\mathbb{R})$ of polynomials with real coefficients of degree less that or equal n is isomorphic to $\mathbb{R}^{n+1}$.
A linear map is injective if its null-space (kernel) $N(T) = \{0\}$. The range, or image, is another subspace associated with T.

If $T : V \to W$ is a map between finite dimensional vector spaces then $\dim(\ker(T) + \dim(im(T)) = \text{di}$

The proof starts with any basis of $\ker(T)$, say $\alpha_1, \cdots, \alpha_k$. Then let $\beta_1, \cdots \beta_l$ be a basis of

$im(T)$. Let $\alpha_{k+1}, \cdots \alpha_{k+l}$ be counter images of the β_j. That is $T(\alpha_{k+1}) = \beta_1, \cdots, T(\alpha_{k+l}) = \beta_l$.
We claim that $\alpha_1, \cdots, \alpha_k, \alpha_{k+1}, \cdots \alpha_{k+l}$ is a basis of V. Indeed, these are linearly independent vectors. Assume that $c_1\alpha_1 + \cdots + c_k\alpha_k + c_{k+1}\alpha_{k+1} + \cdots + c_l\alpha_l = 0$. We apply T on both sides and get $c_{k+1}T(\alpha_{k+1}) + \cdots + c_lT(\alpha_l) = 0, c_{\kappa+1}\beta_1 + \cdots + c_l\beta_l = 0$ Thus $c_{k+1} = \cdots = c_l = 0$ because the $\beta's$ are linearly independent. therefore $c_1\alpha_1 + \cdots + c_k\alpha = 0$ and thus $c_1 = \cdots = c_k = 0$ because $\alpha_1, \cdots, \alpha_k$ are linearly independent.
No let α be any vector in V. Then $T(\alpha) = c_1\beta_1 + \cdots + c_l\beta_l$ because $T(\alpha) \in im(T)$. We look at $\alpha' = c_1\alpha_{k+1} + \cdots + c_l\alpha_{k+l}$. Then $T(\alpha') = T(\alpha)$. This shows $T(\alpha - \alpha') = 0$ or $\alpha - \alpha' \in \ker(T)$. Therefore, $\alpha - \alpha' = d_1\alpha_1 + \cdots + d_k\alpha_k$, and we get
$\alpha = d_1\alpha_1 + \cdots + d_k\alpha_k + c_1\alpha_{k+1} + \cdots + c_l\alpha_{k+l}$
This shows $k + l = n$ which is our claim.

A most surprising consequence is that a linear map on a finite dimensional space is injective if and only if it is surjective.