Relations

Equivalence relations
A relation between elements of a set $A$ and a set $B$ is a subset $R$ of $A \times A$. However, instead of $(a, b) \in R$ we write $aRb$. If $f$ is a function from $A$ to $B$ then $\text{graph}(f) = \{(a, f(a)) | a \in A\}$. We have that for any $a \in A$ there is exactly one $b \in B$ such that $(a, b) \in \text{graph}(f)$. Namely $b = f(a)$. In general we have for a binary relation $R$ many or no $b$ such that $(a, b) \in R$.

A typical example is $A =$ students at your university $B =$ courses offered. Then $(\text{joe smith}, \text{Math 3336}) \in R$ in case that joe smith is enrolled in Math 3336.

A relation $R \subseteq A \times A$ is reflexive if $(a, a) \in R$ for every $a \in A$.

Equality is reflexive: $a = a.$ on every set $A$. Let $\mathbb{Z}$ be the set of integers and $a|b$ be the divisibility relation: $a|b \iff \exists_{c \in \mathbb{Z}} a \cdot c = b$ Divisibility on integers is reflexive. Similarly, $\leq$ on $\mathbb{R}$ is reflexive.

A relation $R \subseteq A \times A$ is symmetric if whenever $(a, b) \in R$ then $(b, a) \in R$ Of course, equality on $A$ is symmetric. The order relation $\leq$ is not symmetric. We can have $a \leq b$ and $b \leq a$ only if $a = b$. Such a relation is called anti-symmetric: $aRb \land bRa \rightarrow a = b$. Divisibility on $\mathbb{N}$ is also anti-symmetric.

A relation $R \subseteq A \times A$ is transitive if $aRb$ and $b Rc$ yields $aRc$. Equality $=$, order $\leq$, and divisibility $|$ are transitive relations on $\mathbb{N}$.

A relation $R \subseteq A \times A$ is an equivalence relation on $A$ if it is reflexive, symmetric and transitive.

Let $f : A \rightarrow B$ be any function from $A$ to $B$. Then
$$\ker(f) = \{(a, a') | f(a) = f(a')\}$$
is an equivalence relation on $A$.

Let $T : U \rightarrow V$ be a linear map between vector spaces. Then
$$N = \ker T = \{a | a \in U, T(a) = 0\}$$

Then $T(a) = T(a') \iff T(a - a') = 0 \iff a - a' \in N \iff a' \in a + N$

For any equivalence relation $R$ on $A$ we define the class of $a$ as
$$[a]_R = \{a'|aRa'\}$$

We note that the equivalence classes for $R$ form a partition $C$ of $A$. That is
a) No class is empty: $a \in [a]_R$.
b) If the intersection of two classes is non-empty then the classes are the same: $c \in [a] \cap [a']$ then $[a] = [a']$
c) the union of all classes is $A : \bigcup_{a \in A} [a]_R = A$

Let $\mathbb{Z}$ be the set of integers and $m > 0$ We defined $a \equiv b$ if $m|a - b$, that is $a - b \in m\mathbb{Z}$. We have $a = q \cdot m + r, 0 \leq r < m$. Thus $a - r = q \cdot m$ or $a \equiv r$ :Every $a \in \mathbb{Z}$ is congruent to some $r$ where $0 \leq r < m$. There are finitely $r$-many classes.

We know that we can add and multiply classes representative wise.
Partial orderings.
A relation $R$ on $A$ is called a partial ordering if it is reflexive, antisymmetric and transitive. The $\leq$ relation on $\mathbb{Z}$ is a partial order. But so is divisibility $|$. $(\mathbb{Z}, \leq)$ and $(\mathbb{Z}, |)$ denote the integers with partial ordering and divisibility, respectively.

Let $A$ be any set. Then $P(A) = \{B | B \subseteq A\}$ is partially ordered: Subsets $B$ of a set $A$ are partially ordered. We write $(P(A), \subseteq)$ is the notation of the powerset of $A$ together with the subset relation.

An element $m_1$ of of a partially ordered set $(A, \leq)$ is called the minimum of $A$ if $m_1 \leq a$ for every $a \in A$. There can be at most one minimum for $(A, \leq)$. The open interval $(0, 1)$ does not have a minimum while 0 is the minimum of $[0, 1)$. Every non-empty subset of natural numbers has a minimum. This is the well-ordering principle for $\mathbb{N}$.

An element $m_2$ is the maximum of $(A, \leq)$ if $a \leq m_2$ for every $a \in A$.

The $\emptyset$ is the minimum of $(P(A), \subseteq)$ and $A$ is the maximum of $(P(A), \subseteq)$.

An element $a$ of $(A, \leq)$ is minimal if there is no element $b$ such that $b < a$. An element $b$ is maximal if there is no element $a > b$.

**Theorem** Let $(S, \leq)$ be a finite partially ordered set. Then $S$ has a minimal element.

Proof. Pick any element $a_1$ from $S$. If $a_1$ is minimal, then we are done. Otherwise, there is some $a_2 < a_1$. If $a_2$ is minimal, we are done. Otherwise there is some $a_3 < a_2$. If $a_3$ is minimal, we are done. Otherwise there is some $a_4 < a_3$. If $a_4$ is minimal, we are done. Otherwise there is some $a_5 < a_4$. This process of creating a sequence $a_1 > a_2 > a_3 > \ldots$ must end because we have only finitely many elements in $S$.

Singletons $\{a\}$ are the minimal elements of $P(A) \setminus \emptyset$, that is the set of all non-empty subsets of $A$. For any $a \in A$ the set $A \setminus \{a\}$ is maximal.

For $(\mathbb{N}, |)$, the set of natural numbers together with divisibility, the prime numbers are minimal for $\mathbb{N} \setminus \{1\}$.

1 is the minimum of $(\mathbb{N}, |)$ while 0 is the maximum.

Let $B$ be a subset of $(A, \leq)$. An element $c$ is called an upper bound of $B$ if $b \leq c$ for every $b \in B$. An upper bound which belongs to $B$ must be the maximum of $B$. The open interval $(0, 1)$ has every number $c \geq 1$ as an upper bound. If the set of upper bounds has a minimum then there is a least upper bound.

Every set non-empty $S$ of real numbers which has an upper bound has a least upper bound. This is an axiom for real numbers.

A partially ordered set $(L, \leq)$ is called a lattice any two element subset $\{a, b\}$ has a least upper bound, $a \lor b$, and a largest lower bound $a \land b$. That is, if $c \geq a$ and $c \geq b$ then $c \geq a \lor b$. Similarly, if $d \leq a$ and $d \leq b$ then $d \leq a \land b$.

The set $\mathbb{N}$ of natural numbers together with divisibility is a lattice
\[ n \land m = \gcd(n, m) = (n, m), n \lor m = \lcm(n, m) = [n, m] \]
Any finite partially ordered set can be totally ordered. This process is called "topological sorting". Let \((A, \leq)\) be given where \(A\) is finite. We choose a minimal \(a_0\) of \(A\). This will be the minimum of \(A\). Then choose a minimal element \(a_1\) of \(A\setminus\{a_0\}\) and set \(a_0 \leq a_1\). Actually, we may assume that \(A\setminus\{a_0\}\) has been totally ordered \(a_1 \leq a_2 \leq \ldots \leq a_n\) such that if \(a_i \leq a_j\) then \(a_i \leq a_j\).