Relations

Equivalence relations

A relation between elements of a set A and a set B is a subset R of $A \times A$. However, instead of $(a,b) \in R$ we write aRb. If f is a function from A to B then $graph(f) = \{(a,f(a))|a \in A\}$. We have that for any $a \in A$ there is exactly one $b \in B$ such that $(a,b) \in graph(f)$. Namely $b = f(a)$. In general we have for a binary relation R many or no b such that $(a,b) \in R$.

A typical example is $A =$ students at your university $B =$ courses offered. Then $(joe\ smith, Math\ 3336) \in R$ in case that $joe\ smith$ is enrolled in $Math\ 3336$.

A relation $R \subseteq A \times A$ is *reflexive* if $(a,a) \in R$ for every $a \in A$.

Equality is reflexive: $a = a$. on every set A. Let $\mathbb{Z}$ be the set of integers and $a|b$ be the divisibility relation: $a|b \Leftrightarrow \exists_{c\in\mathbb{Z}}\ a \cdot c = b$ Divisibility on integers is reflexive. Similarly, $\leq$ on $\mathbb{R}$ is reflexive.

A relation $R \subseteq A \times A$ is *symmetric If* whenever $(a,b) \in R$ then $(b,a) \in R$ Of course, equality on A is symmetric. The order relation $\leq$ is not symmetric. We can have $a \leq b$ and $b \leq a$ only if $a = b$. Such a relation is called *anti-symmetric* : $aRb \wedge bRa \to a = b$. Divisibility on $\mathbb{N}$ is also anti-symmetric.

A relation $R \subseteq A \times A$ is *transitive* if aRb and bRc yields aRc. Equality $=$, order $\leq$, and divisibility $|$ are transitive relations on $\mathbb{N}$.

A relation $R \subseteq A \times A$ is an *equivalence* relation on A if it is reflexive, symmetric and transitive.

Let $f : A \to B$ be any function from A to B. Then

$$\ker(f) = \{(a,a')|fa) = f(a')\}$$

is an equivalence relation on A.

Let $T : U \to V$ be a linear map between vector spaces. Then

$$N = \ker T = \{\alpha|\alpha \in U, T(\alpha) = 0\}$$

Then $T(\alpha) = T(\alpha')$ iff $T(\alpha - \alpha') = 0$ iff $\alpha - \alpha' \in N$ iff $\alpha' \in \alpha + N$

For any equivalence relation R on A we define the class of a as

$$[a]_R = \{a'|aRa'\}$$

We note that the equivalence classes for R form a partition C of A. That is

a) No class is empty: $a \in [a]_r$.

b) If the intersection of two classes is non-empty then the classes are the same: $c \in [a] \cap [a']$ then $[a] = [a']$

c) the union of all classes is A : $\cup_{a\in A} [a]_R = A$

Let $\mathbb{Z}$ be the set of integers and $m > 0$ We defined $a \equiv b$ if $m|a - b$, that is $a - b \in m\mathbb{Z}$. We have $a = q \cdot m + r, 0 \leq r < m$. Thus $a - r = q \cdot m$ or $a \equiv r$:Every $a \in \mathbb{Z}$ is congruent to some r where $0 \leq r < m$. There are finitely r–many classes.

We know that we can add and multiply classes representative wise.

Partial orderings.

A relation R on A is called a *partial ordering* if it is reflexive, antisymmetric and transitive. The $\leq$ relation on $\mathbb{Z}$ is a partial order. But so is divisibility $|$. $(\mathbb{Z}.\leq)$ and $(\mathbb{Z},|)$ denote the integers with partial ordering and divisibility, respectively.

Let A be any set. Then $P(A) = \{B|B \subseteq A\}$ is partially ordered: Subsets B of a set A are partially ordered. We write $(P(A),\subseteq)$ is the notation of the powerset of A together with the subset relation.

An element m_1 of of a partially ordered set $(A,\leq)$ is called the *minimum* of A if $m_1 \leq a$ for every $a \in A$. There can be at most one minimum for $(A,\leq)$. The open interval $(0,1)$ does not have a minimum while 0 is the minimum of $[0,1)$. Every non-empty subset of natural numbers has a minimum. This is the well-ordering principle for $\mathbb{N}$.

An element m_2 is the *maximum* of $(A,\leq)$ if $a \leq m_2$ for every $a \in A$.

The $\emptyset$ is the minimum of $(P(A),\subseteq)$ and A is the maximum of $(P(A),\subseteq)$.

An element a of $(A,\leq)$ is *minimal* if there is no element b such that $b < a$. An element b is maximal if there is no element $a > b$.

Theorem Let $(S,\leq)$ be a finite partially ordered set. Then S has a minimal element.

Proof. Pick any element a_1 from S. If a_1 is minimal, then we are done. Otherwise, there is some $a_2 < a_1$. If a_2 is minimal, we are done. Otherwise there is some $a_3 < a_2$. If a_3 is minimal, we are done. Otherwise there is some $a_4 < a_3$. If a_4 is minimal, we are done. Otherwise there is some $a_5 < a_4$. This process of creating a sequence $a_1 > a_2 > a_3 > \ldots$ must end because we have only finitely many elements in S.

Singletons $\{a\}$ are the minimal elements of $P(A)\backslash\emptyset$, that is the set of all non-empty subsets of A. For any $a \in A$ the set $A\backslash\{a\}$ is maximal.

For $(\mathbb{N},|)$, the set of natural numbers together with divisibility, the prime numbers are minimal for $\mathbb{N}\backslash\{1\}$.

1 is the minimum of $(\mathbb{N},|)$ while 0 is the maximum.

Let B be a subset of $(A,\leq)$. An element c is called an *upper bound* of B if $b \leq c$ for every $b \in B$. An upper bound which belongs to B must be the maximum of B. The open interval $(0,1)$ has every number $c \geqq 1$ as an upper bound. If the set of upper bounds has a minimum then there is a *least upper bound.*

Every set non-empty S of real numbers which has an upper bound has a least upper bound. This is an axiom for real numbers.

A partially ordered set $(L,\leq)$ is called a *lattice* any two element subset $\{a,b\}$ has a least upper bound, $a \vee b$, and a largest lower bound $a \wedge b$. That is, if $c \geq a$ and $c \geq b$ then $c \geq a \vee b$. Similarly, if $d \leq a$ and $d \leq b$ then $d \leq a \wedge b$.

The set $\mathbb{N}$ of natural numbers together with divisibility is a lattice

$$n \wedge m = \gcd(n,m) = (n,m), n \vee m = \operatorname{lcm}(n,m) = [n,m]$$

Any finite partially ordered set can be totally ordered. This process is called "topological sorting". Let $(A, \preceq)$ be given where A is finite. We choose a minimal a_0 of A. This will be the minimum of A. Then choose a minimal element a_1 of $A \backslash \{a_o\}$ and set $a_0 \leq a_1$. Actually, we may assume that $A \backslash \{a_o\}$ has been totally ordered $a_1 \leq a_2 \leq \ldots \leq a_n$ such that if $a_i \preceq a_j$ then $a_i \leq a_j$.