Predicates

The propositional calculus is limited in its expressive power. You cannot express relations between elements, for example the order relation between numbers. while $2 < 3$ is a statement which is a proposition, for every number x there is a number y such that $x < y$ is a true statement but cannot be formulated within the propositional calculus. One needs predicates, or relations for that. In order to set up the language of the predicate calculus one start with atomic formulas $r(x), s(x,y)$, $t(x,y,z)$ etc in $one, two, three$ variables x, y, xyz or more generaly relational symbols $r(x_1, \cdots, x_n)$ in n variables. For example, $s(x,y,z)$ may be ternary relation between numbers with the meaning that $s(x,y,z)$ hols on $\mathbb{N}$ in case that $x+z = z$ is true. For example $s(1,2,3)$ holds because $1+2 = 3$. But also $s(2,5,7)$ is true on $\mathbb{N}$.

We may assume that we have for every k an infinite list of relational symbols in k variables:

$r_1(x), r_2(x), \ldots, r_n(x), \ldots; r_1(x,y), r_2(x,y), \ldots, r_n(x,y); r_1(x,y,z), r_2(x,y,z), \ldots; \ldots; r_1(x_1, x_2, \ldots$

These are the *atomic* formulas. If α and β are formulas then the propositional combinations

$$\neg\alpha, (\alpha \wedge \beta), (\alpha \vee \beta), (\alpha \rightarrow \beta)$$

are formulas. But we also have for the predicate calculus an infinite list of variables $x_1, x_2, \ldots, x_n, \ldots$ which we need for quantifications:

$$\forall_{x_n}\alpha, \exists_{x_n}\alpha$$

A first order formula is obtained from atomic formulas by finitely many applications of the propositional connectives and quantifications.

$r(x,y), s(x,y,z),$ are atomic formulas. $(r(x,y) \wedge \forall_z r(x,z))$ is a first order formula

Given a specific domain D for the variables x_n then the truth of atomic formulas is given if the variables are interpreted by elements of D.

The variables of an atomic formula are all **free**. Assume that we know the sets $v(\alpha)$ and $v(\beta)$ of free variables of α and β. Then the set of free variables of $(\alpha \wedge \beta)$ is $v(\alpha) \cup v(\beta)$. similar for the other propositional variables. However, $v(\forall_{x_n}\alpha) = v(\alpha) \backslash \{x_n\}$ if x_n occurs in a, otherwise it is $v(\alpha)$.

A formula which doe not have any free variables is called a sentence. For a specific domain, a sentence has a unique truth value.

Quite popular in logic and computer science is reverse Polish notations which does not need parentheses:

From atomic formulas you form

$$\neg\alpha, \wedge\alpha\beta, \vee\alpha\beta, \rightarrow \alpha\beta, \forall_{x_n}a, \exists_{x_n}\alpha$$

to get after finitely many applications all formulas.

For example,

$$\forall x \wedge r(x,y)s(y,z,z)$$

Is the universal quantification of the conjunction of $r(x,y)$ and $s(y,z,z)$.

We usually prefer the traditional way of writing formulas, that is with parentheses.

Of great importance are the **De Morgan's laws for quantifiers**

$$\neg\forall x P(x) \equiv \exists x \neg P(x), \neg\exists x P(x) \equiv \forall x \neg P(x)$$

$P(x)$ is a formula which may or may not involve the variable x which ranges over a specific domain D.

We mcan easily see that $\forall x(P(x) \wedge Q(x)) \ \equiv \forall x P(x) \wedge \forall x Q(x)$ while $\forall x(P(x) \vee Q(x))$ is not equivalent to $\forall x P(x) \vee \forall x Q(x)$

$\forall x(P(x) \wedge Q(x))$ says that for every element x of the domain $P(x)$ as well as $Q(x)$ are true, which is $\forall x P(x) \wedge \forall x Q(x)$. In order to show that $\forall x P(x) \vee \forall x Q(x)$ is not equivalent to $\forall x P(x) \vee \forall x Q(x)$ let D be the domain of natural numbers and $P(x)$ be the predicate x is even and $Q(x)$ be x is odd. Then $\forall x P(x) \vee \forall x Q(x)$ is true but $\forall x P(x) \vee \forall x Q(x)$ is not.

Similar, $\forall x(P(x) \rightarrow Q(x))$ is not equivalent to $\forall x P(x) \rightarrow \forall x Qx$.