Name: **May 7, 2011**

FINAL

Math 5330 Abstract Algebra Online

This Final is worth 200 points. You are not allowed to use any books or notes. The last two problems are optional and each is worth 20 extra points.
You have three hours to complete the test.

1. Say what a group $(G, \cdot)$ is and give two examples for groups, where one is commutative and the other is not. **Answer**: The operation $\cdot$ is associative, $a \cdot (b \cdot c) = (a \cdot b) \cdot c$, there is a unit e, such that $a \cdot e = e \cdot a = a$ for every $a \in G$, and every $a \in G$ has an inverse a^* such that $a \cdot a^* = a^* \cdot a = e$. The inverse of a is denoted as a^{-1}. The reals with addition is a commutative group, the group of permutations of an n –elements set is not commutative if $3 \leq n$.
2. **a**. Define that H is a subgroup of G. **Answer**: H is a subset which contains the unit e, is closed under multiplication and taking inverses.
   b. Define that N is a normal subgroup of G. **Answer**: if $a \in N$ then for every $g \in G$ one has that $gag^{-1} \in N$.
3. Prove that for the additive group $(\mathbb{Z}, +)$ of integers every subgroup is of the form $k\mathbb{Z}$. **Answer**: Let H be a subgroup of $\mathbb{Z}$. Let k be the smallest natural number greate than 0 which belongs to H. Let $h \in H$. Then divide h by k withe remainder: $h = qk + r, 0 \leq r < k$. Because $k \in H, h \in H$ we get $r \in H$. But then $r = 0$(why?) and therefore $h = qk \in k\mathbb{Z}$. That $k\mathbb{Z} \subseteq H$ is clear because $k \in \mathbb{Z}$.
4. Let G be a finite group of order n and let H be a subgroup of order m. Outline a proof that $m|n$. **Answer**: This is Lagrange's Theorem. Read the proof.
5. Let H and K be both subgroups of the group G. Assume that the orders of H and K are relatively prime. Prove that $H \cap K = \{e\}$. **Answer**: $H \cap K$ is a subgroup of H as well as K. Its order is a divisor of the order of as well as of K which are relatively prime. So it must be 1
6. Let N be a normal subgroup of the group G. How is the factor group G/N defined? Say what its elements are and how multiplication is defined. What is the unit of the group G/N? **Answer**: All in the book!
7. **a**. Define that ϕ is a homomorphism from the group G to the group H. How are isomorphisms defined?Answer: $\phi(ab) = \phi(a)\phi(b), \phi(a^{-1}) = \phi(a)^{-1}, \phi(e) = e.$ An isomorphism is injective and surjective
   b. Let ϕ be a homomorphism from the group G to the group H. How is the kernel N of ϕ defined? **Answer**: $N = \{g|\phi(g) = e\}$
8. Let$G$ and H be finite groups and let $G \to H$ be a surjective homomorphism. Show that the order $|H|$ of H divides the order $|G|$ of G. **Answer**: By the fundamental theorem we have $G/N \cong H.$ Therefore $o(G/N) = o(H)$ and $o(N)o(H) = o(G)$
9. Let $G =< x >$ be a finite cyclic group of order n. Prove that G is isomorphic to the group $\mathbb{Z}_n$ of integers modulo n. Answer: $\phi : k \mapsto x^k$ is homomorphic. We have $x^k = e$ iff $n|k$.

Therefore $\ker(\phi) = n\mathbb{Z}$. By the fundamental theorem, $\mathbb{Z}/nz \cong G$

10. Define the direct product of two groups G and H. Give examples where the direct product $\mathsf{Z}_n \times \mathsf{Z}_m$ is cyclic, and where it is not cyclic. **Answer**: $\mathbb{Z}_{2\times}\mathbb{Z}_3 \cong \mathbb{Z}_6$ is cyclic. $\mathbb{Z}_{2\times}\mathbb{Z}_2 \cong V$, Klein's Vierer group is not cyclic.

11. Let a and b be elements of the principal ideal domain D. Prove that $(a,b) = \{xa + yb \mid x,y \in D\}$ is an ideal. Then if d is the generator of (a,b) prove that

- **a**. $d|a$ and $d|b$;
- **b**. If $c|a$ and $c|b$ then $c|d$. That is d is the greatest common divisor of a and b. **Answer**: Was a last homework problem.

12. Let $H = \{4n + 1 | n \in \mathsf{N}\}$, that is, $H = \{1, 5, 9, 13, 17, 21, 25, \ldots\}$. Call a number in H *irreducible* if it is different from 1 and **not** the product of two smaller numbers which belong to H. Of course, all prime numbers that belong to H are irreducible. For example, $5, 13, 17, \ldots$ are irreducible because they are prime numbers, but also the numbers $9, 21, \ldots$ are irreducible because you cannot factor them using only numbers that belong to H. Now prove:

- **a**. Every number in H is a product of irreducible numbers from H.
- **b**. Give an example for a number in H where the factorization into a product of irreducible numbers is not unique.