Math 5330 Abstract Algebra
March 23, 2019
Test 2

You have 90 minutes to complete the test. You cannot use any books, calculators or phones.

1. What is the order of 2 in
a. $\mathbb{Z}_4$ **Solution** $o(2) = 2$
b. $\mathbb{Z}_5$ **Solution** $o(2) = 5$
c. $\mathbb{Z}_6$ **Solution** $o(2) = 3$
d. $\mathbb{Z}_7$ **Solution** $o(2) = 7$
2. Is $\mathbb{Z} \times \mathbb{Z}_2$ is cyclic? You must prove your answer. **Solution**: No! $(1,1)$ could be a generator but it is not: $(2,1) \notin < (1,1) >$
3. Define that
a. $f : A \to B$ is an injective function. .
b. $f : A \to B$ is a surjective function.
c. Can you find an in injective function $\mathbb{N} \to \mathbb{N} \times \mathbb{N}$? If your answer is "Yes" then you must give an example. If your answer is "No" then you must provide a reason. **Solution**:Yes, $f(n) = (n,n)$ is injective.
d. Can you find an injective function $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$? If your answer is "Yes" then you must provide an example. If your answer is "No" then you must provide a reason. **Solution**; Yes.
4. Prove that a subgroup of a cyclic group is cyclic. **Solution**: Let $G =< x >$ and H be a subgroup of G. Then let k be the smallest positive number such that $x^k \in H$. Then assume $x^n \in H$. By the division algotithm, $n = qk + r$ where $0 \le r < k$. We have that $x^n = x^{qk}x^r$, therefore $x^{-qk}x^n = x^r$ which shows $x^r \in H$ By the choice of k we get $r = 0$. Thus $x^n = (x^k)^q$. More detailed in the book!
5. Find the right cosets of $< 3 >$, the cyclic group generated by 3, for
a. $\mathbb{Z}_{12}$. **Solution**: $H =< 3 >$ has four elements $H = \{0,3,6,9\}$ and cosets are $H, H+1, H+2, H+3$
b. $\mathbb{Z}$ **Solution**: $< 3 >= 3\mathbb{Z}$, cosets are $3\mathbb{Z} + r, r = 0,1,2$
6. Define that E is an equivalence relation on A.
a. For $a \in A$ define the equivalence class $[a]_{E.}$.
b. Show that two different equivalence classes are disjoint.
7. Let $f : A \to B$ be any map. Prove that
a. $E_f = \{(a_1,a_2) | f(a_1) = f(a_2)\}$ is an equivalence relation on A.
b. For the map $^2 : \mathbb{R} \to \mathbb{R}, x \mapsto x^2$ describe the equivalence classes. **Solution**: $\{x,-x\}, x \in \mathbb{R}^+$
8. Let H be a subgroup of the group G. Define a binary relation $\equiv$ on G by $x \equiv y$ iff $x^{-1}y \in H$. **Solution**: This is the decomposition into right cosets.
b. What is the equivlence class of e? Prove that $[x] = xH = \{xh | h \in H\}$.

9. Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & 5 & 10 & 8 & 2 & 9 & 4 & 7 & 1 & 6 \end{pmatrix}$. Decompose σ into cycles and then transpositions. Is σ even or odd. **Solution**:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & 5 & 10 & 8 & 2 & 9 & 4 & 7 & 1 & 6 \end{pmatrix} = (1,3,10,6,9)(2,5)(4,8,7) = (1,9)(1,6)(1,10)(1,3)($$

odd

10. Let A_n be the set of all even permutations of S_n. Prove that A_n is a subgroup of S_n. **Solution**: The identity permutation is even, e.g., id$=(1,2)(1,2)$; if σ_1 and σ_2 are both even then σ_1 is a product of an even number of transpositions ans σ_2 is an even number of transpositions and so is $\sigma_1 \circ \sigma_2$; if $\sigma = \tau_1 \circ \tau_2 \circ \ldots \circ \tau_k, k$ even, then $\sigma^{-1} = \tau_k \circ \ldots \tau_2 \circ \tau_1$ is also even. Notice $\tau = \tau^{-1}$ for transpositions.