Math 3336 Test 4 April 17, 2018

Each problem is worth 20 **points. You cannot use any calculators, books or notes.**

1. State the well-ordering principle for natural numbers. **Answer**: Every non-empty set of natural numbers has a minimum.
2. State the induction principle for natural numbers. **Answer**: a) Assume that a poperty P holds for $n = 1.$ b) If P ho;ds for n then it holds for $n+1.$ From a) the induction basis and b) the induction step one can conclude that P holds for all natural numbers.
3. Prove the induction principle from the well-ordering principle. **Answer**: Assume that there is a natural number for which P doesn'thold. Let $S = \{n|$ P does not hold for $n\}.$ Then S is not empty and must have a smallest element m. Because P holds for $n = 1,$ we have that $m > 1.$ Therefore m has a predecessor $m-1.$ It must be the case that P holds for $m-1$ because m is the smallest number for which P does not hold. But then P must hold for $(m-1)+1 = m.$ This is a contradiction. S ust be empty, that is P holds for all natural numbers.
4. State strong induction. **Answer**: Assume a) Property P holds for $n = 1$ b) One has that P holds for n in case P holds for all $k < m.$ Then P holds for every natural number.
5. Prove that every natural number is a product of prime numbers. **Answer**: If n is prime, then we are done. Otherwise $n = a \cdot b$ where a as well as b are smaller that $n.$ If we assume that every number smaller than n is a product of primes then we have $a = p_1 \cdot p_2 \ldots \cdot p_k$ and $b = q_1 \cdot q_2 \ldots \cdot q_l$ are products of primes. But then $a \cdot b = (p_1 \cdot p_2 \ldots \cdot p_k) \cdot (q_1 \cdot q_2 \ldots \cdot q_l)$ is a product of primes. By strong induction, every number is a product of primes. The number 1 is a product of primes, the empty product. Also 2 is a product of primes, 2 is a prime.
6. Let p be a prime number. Assume that $p|a \cdot b.$ Prove that $p|a$ or $p|b.$ **Answer**: Assume that the prime number $p \not| \; a.$ Then $\gcd(p,a) = 1.$ Therefore1 is an integer comination of p and $a.$ Thus $1 = u \cdot a + v \cdot p.$ Multiply both sides by b and we get $b = u \cdot a \cdot b + v \cdot p \cdot b.$ We have $p|u \cdot a \cdot b$ because we assume that $p|a \cdot b$ and $p|v \cdot p \cdot b,$ obviously. We conclude that $p|b.$
7. Outline a proof that every natural number is a unique product of primes. **Answer**: Assume $n = p_1 \cdot p_2 \ldots \cdot p_k = q_1 \cdot q_2 \ldots \cdot q_l.$ We see that $p_1 | q_1 \cdot (q_2 \ldots \cdot q_l)$ By the previous problem, $p_1|q_1$ or $p_1|(q_2 \cdot \ldots \cdot q_l)$ that is $p_1|q_1$ or $p_1|q_2$or $p_1|(q_3 \cdot \ldots \cdot q_l)$ we get that p_1 divides one of the $q_i.$ We nay assume that p_1 divides $q_1.$ Because q_1 is prime, we get $p_1 = q_1$. We cancel p_1 on both sides. Assuming uniques prime factorization for all natural numbers smaller than n we are done.
8. Prove by mathematical induction.

   a. $1 + 2 + \cdots n = \frac{n(n+1)}{2}$ **Answer**: For $n = 1$the formula says

$1 = \frac{1\cdot(1+1)}{2} = 1;$ assume that $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$. Then we get for $n+1$ that $1 + 2 + +n + (n+1) = \frac{n(n+1)}{2} + (n+1) = \frac{n(n+1)+2(n+1)}{2} = \frac{(n+1)(n+2)}{2}$ which is the formula for $n+1$.

b. $1 + 3 + 5 + \ldots + (2n+1) = (n+1)^2$ **Answer**: For $n = 1$ the formula says $1 + 3 = (1+1)^2 = 2^2 = 4$ which is correct. Assume now that one has for some n that $1 + 3 + 5 + \ldots + (2n+1) = (n+1)^2$. Then $1 + 3 + 5 + \ldots + (2n+1) + (2n+3) = (n+1)^2 + (2n+3) = n^2 + 2n + 1 + 2n + 3 =$ which shows the formula for $n+1$.

c. Any finite set which has $n - many$ elements has $2^n - many$ subsets. **Answer**: Any one element set $S = \{1\}$ has two subsets, the emty set Ø and the whole set $\{1\}$. Assume that the set $S = \{1, 2, \ldots, n\}$ of $n - many$ elements has $2^n - many$ subsets. The subsets of $S^+ = \{1, 2, \ldots, n, n+1\}$ fall into two equal numbered groups: subsets which contain the element $n+1$ and those which don't contain $n+1$. Thus there are $2 \cdot 2^n = 2^{n+1} - many$ subsets of S^+.

8. Give a recursive definition of the sum $S_n = 1 + 2 + \cdots + n$ and of the product $P_n = 1 \cdot 2 \cdot \ldots \cdot n$ of the first n numbers. **Answer**:
$S_1 = 1, S_{n+1} = S_n + (n+1); P_n = 1, P_{n+1} = P_n \cdot (n+1)$

10. Show that the set S defined by $5 \in S$, and $x + y \in S$ whenever $x \in S$ and $y \in S$ is the set of all positive integers that are multiples of 5. Hint: You need to show that S contains all number that are multiple of 5. You do this by induction over n.

then you use that every number $z \in S$ diferent from 5 is of the form $z = x + y$. **Answer**: We first show that $5 \cdot \mathbb{N} \subseteq S$. We show this by induction. We have $5 \cdot 1 = 5 \in S$. Assume that $5n \in S$. But then $5n + 5 = 5(n+1) \in s$. By induction we get $5 \cdot n \in S$ for every n, that is $5 \cdot \mathbb{N} \subseteq S$. Now we need to show that $S \subseteq 5 \cdot \mathbb{N}$. This is true for $5 \in S$. Now, if x as well as y are divisble by 5, then $x + y$ is divisble by 5. Thus every element of S is divisible by 5.