September 12, 2014

Homework 2 Math 6302

1. Let A be any set. We assume that all of its elements a are sets. Show that the set $R(A) = \{a|a \in A, a \notin a\}$ is not an element of A.

2. For any set A, let B be the set of all second coordinates of ordered pairs that belong to A. Then let $C = \{(a, R(B))|a \in A\}$. Show that $A \cap C = \emptyset$. Hence, for any set A there is a set C that is disjoint and equipotent to A.

3. An injective φ map from the partially ordered set $(P, \leq)$ to the partially ordered set $(Q, \leq)$ is called an order embedding if $\varphi(x) \leq \varphi(y)$ iff $x \leq y$. A bijective order embedding is an order isomorphism. That is, φ^{-1} is monotone. Prove: Any ordered set can be embedded into an ordered set that has a minimum and a maximum.

4. Any partial order on a finite set can be extended to a total order. This "process" is called topological sorting.

5. Assume that every subset of the partially ordered set $(L, \leq)$ has an infimum. Prove that every subset has a supremum.

6. A reflexive and transitive relation R on a set S is called a *quasi-order*. Show that

   $$a \sim_R b \Leftrightarrow a\ R\ b \text{ and } b\ R\ a$$

   is an equivalence relation on S. On $S/\sim_R$ a partial order $\leq_R$ can be defined

   $$[s] \leq [t] \text{ iff } sRt$$

   $(S, \leq_R)$ is called the *contraction* of the quasiorder R.

7. Let $f : (S, R) \twoheadrightarrow (S/\sim_R, \leqslant)$, $s \mapsto [s]$, be the contraction-map of the quasi-order (S, R) . Then f is increasing , i.e., $s_1 R s_2 \Rightarrow [s_1] \leqslant [s_2]$, and any increasing map $h : (S, R) \to (T, \leqslant)$ from (S, R) into an ordered set $(T, \leqslant)$ factors over f, i.e., $h = g \circ f$ for a **uniquely determined** increasing map $g : (S/\sim_R, \leqslant) \to (T, \leqslant)$. One says, $f : (S, R) \twoheadrightarrow (S/\sim_R, \leqslant)$ is *universal* for all increasing maps $h : (S, R) \to (T, \leqslant)$. Show: If $f' : (S, R) \to (C, \leqslant)$ is also universal for all increasing maps from (S, R) into ordered sets $(T, \leqslant)$, then one has that $(S/_R, \leqslant) \cong (C, \leqslant)$.