January 30, 2012

Homework 1 Math 6303

1. Let $p(x)$ be an irreducible polynomial in $\mathbb{F}[x]$ where $\mathbb{F}$ is a field.

   (a) Show that $\mathbb{F} \to \mathbb{F}[x]/(p(x)), a \mapsto [a]$ is an injective homomorphism (embedding) between fields. Thus, we can think of $\mathbb{F}[x]/(p(x))$ as an extension $\mathbb{K}$ of $\mathbb{F}$.

   (b) Show that $p(x)$ has a root in $\mathbb{K}$.

   (c) Show that for every polynomial $p(x)$ of degree n with coefficients of a field $\mathbb{F}$ admits a field extension $\mathbb{K}$ where $p(x)$ has n roots.

2. Explain why $\mathbb{R}[x]/(x^2+1)$ is isomorphic to the complex numbers.

3. Let the field $\mathbb{E}$ be an extension of the field $\mathbb{F}$. Then $\mathbb{E}$ can be perceived as a vector space over $\mathbb{F}$. The dimension of $\mathbb{E}$ over $\mathbb{F}$ is called the *degree*$[\mathbb{E} : \mathbb{F}]$ of the extension $\mathbb{E}|\mathbb{F}$. An extension $\mathbb{E}$ over $\mathbb{F}$ is called *finite* if the degree $[\mathbb{E} : \mathbb{F}]$ is finite. An element α of $\mathbb{E}$ is called *algebraic* if it is a root of a polynomial in $\mathbb{F}[x]$. Otherwise it is called *transcendental*.

   Let M be a subset of $\mathbb{E}$. Then $\mathbb{F}(M)$ denotes the smallest subfield of $\mathbb{E}$ which contains M and $\mathbb{F}$.

   An extension $\mathbb{E}$ of $\mathbb{F}$ is *simple* if $\mathbb{E} = \mathbb{F}(\alpha)$

   (a) Any α in $\mathbb{E}$ gives rise to a homomorphism $\mathbb{F}[x] \to \mathbb{E}, f(x) \mapsto f(\alpha)$. If α is algebraic then one has that $\mathbb{F}(\alpha) \cong \mathbb{F}[x]/(p(x))$ where $p(x)$ is a uniquely determined monic irreducible polynomial. The degree of $p(x)$ is the degree of the extension. If α is transcendental then $\mathbb{F}(\alpha) \cong \mathbb{F}[x]$

   (b) Let $\mathbb{E} = \mathbb{F}(\alpha)$ where α is algebraic. Then for every element β there is a polynomial $r(x)$ of degree less than n where n is the degree of $\mathbb{E}$ over $\mathbb{F}$ such that $\beta = r(\alpha)$.

   (c) Assume that $\mathbb{K}$ over $\mathbb{E}$ and $\mathbb{E}$ over $\mathbb{F}$ are finite. Then $\mathbb{K}$ over $\mathbb{F}$ is finite and one has the *degree rule*:
   $$[\mathbb{K} : \mathbb{F}] = [\mathbb{K} : \mathbb{E}] \cdot [\mathbb{E} : \mathbb{F}]$$

   (d) Prove that the algebraic elements of $\mathbb{E}$ over $\mathbb{F}$ form a subfield of $\mathbb{E}$ which contains $\mathbb{F}$.