Problems and Comments on Section 1

Problems: 1.3, 1.9

For the time being, you can skip section 5. This example will be better understood after we have covered rings (Section 16, Exercises 16.16, 16.17):

For any set X one has that the powerset $P(X)$ of X is a *boolean ring* where addition is the symmetric difference: $+ = \bigtriangledown$ and multiplication is the intersection: $\cdot = \cap$, zero is the empty set: $0 = \emptyset$ and one is the whole set $1 = X$.

Comments:

The text defines only *binary* operations*.* An *n-ary operation* is a map

$$f : A^n \to A$$

that is for every *n-tuple* of elements from A a unique element $f(a_1, a_2, \ldots, a_n) \in A$ is assigned as operation value for f. A *unary operation* assignes to every element of A a uniquely determined element of A. That is, a unary operation is just a map on A. Taking the additive inverse of an integer is an example of a unary operation:

$$-: \mathbb{Z} \to \mathbb{Z}, n \mapsto -n$$

Now, what is a *nullary operation?* Well, we have to say what A^0 should be. It is meaningful to define A^0 as 1. Recall that 0 is defined as the empty set, $0 = \emptyset$ and 1 as the set which contains only zero, that is the empy set: $1 = \{\emptyset\}$. Hence,

$$A^0 = \{0\}$$

and a nullary operation then is a map

$$c : A^0 \to A, 0 \mapsto a$$

which assigns to 0 a unique element in A. Such a map is called a *constant*. We can also think that an *n-ary* operation depends on *n-many* arguments. A *nullary* operation depends on *zero many* arguments, it is constant.

Examples for constants are: The *zero* in the additive structure of the integers. For this algebra we write:

$$\mathbb{Z} = (Z, +, -, 0)$$

making a notational distinction between the set Z of integers and the algebraic structure on that set. This structure makes $\mathbb{Z}$ to what we will learn as a *group*.

A *universal algebra* is a set A together with a family f_i of operations, each of an arity n_i :

$$\mathbf{A} = (A, (f_i)_{i \in I})$$

For the integers we can choose $I = \{0, 1, 2\}$ and $f_0 = +$, $f_1 = -$, $f_2 = 0$. Of course, we will soon extend the algebra of integers to include also multiplication. The multiplicative unit is the constant 1:

$$\mathbb{Z} = (Z, +, -, 0, \cdot, 1)$$

is what is called the *ring* of integers with addition and multiplication.