Problems and Comments on Induction
Chapter 4

Section 4.1, Problems: 25, 32, 35, 47

Comments. We will take the following for granted: Let \( S \) be a non-empty subset of natural numbers. Then \( S \) contains a smallest element. This is called the well-ordering principle. The argument for showing this principle is clear. Let \( n \) be any element in \( S \). Because \( S \) is non-empty, there must be such an \( n \). If \( n \) is already the smallest element in \( S \), we are done. Otherwise, there is a smaller element \( n_1 \) in \( S \). If \( n_1 \) is the smallest element in \( S \), we are done. Otherwise there is a smaller element \( n_2 \) in \( S \). Because we cannot have an infinite descending chain \( n > n_1 > n_2 > n_3 > \cdots \) of natural numbers smaller than \( n \), we must arrive this way at the smallest number in \( S \).

From the well-ordering principle we can deduce the proof principle of Mathematical Induction. In order to prove a statement about natural numbers, \( P(n) \), it is enough to prove \( P(0) \), which is the basis step, together with the inductive step, which is the implication \( P(n) \rightarrow P(n+1) \). Indeed, if we had some \( n \) for which \( P \) would not be true, then the set \( S = \{ n \mid \neg P(n) \} \) would be non-empty. Thus \( S \) would have a least element, \( m \). This \( m \) cannot be 1, because \( P \) is true for 1. Thus \( m \) must have a predecessor, \( m - 1 \), which is a natural number. But \( P(m - 1) \) is true. We have already chosen as number \( m \) the smallest number for which \( P \) is not true, and \( m - 1 \) is smaller than \( m \). But then the inductive step: \( P(m - 1) \rightarrow P(m) \) yields that \( P(m) \) must hold. But this is a contradiction, \( P \) does not hold for \( m \).

Example 11, p. 247, is a beautiful and non-trivial example of mathematical induction. There is a second version of induction. Assume that we can show the following: \( P(1) \) holds and \( P(n) \) holds, in case that \( P(k) \) holds for every \( k < n \). Then \( P \) holds for all natural numbers \( n \). Indeed, assume that we had a number \( n \) for which \( P \) does not hold. We take the smallest such number, \( n \). It cannot be 1. But by the choice of \( m \), we have \( P(k) \) for all \( k < n \). But then \( P(n) \) holds, which is a contradiction.

This second principle of complete induction is often used in algebra. For example in order to show that every natural number is a product of primes. We define 1 as the empty product of primes. Then, if \( n \) is any natural number, it is either a prime, and we are done, or it is the product of two smaller numbers \( n_1 \) and \( n_2 \). Assuming that every number smaller than \( n \) is a product of primes, \( n_1 \) as well as \( n_2 \) are products of primes. But then \( n = n_1 \cdot n_2 \) is a product of primes.