Problems and Comments For Section 0

Problems: 0.7, 0.18; 0.16

Not all problems on Induction require induction. 0.16 is better proven on the basis of the *division algorithm* (p. 20, Lemma 2.1). Of course this algorithm is known to you from elementary school: If n is any integer and d a *positive integer* then d goes a certain number of times evenly into n, say q many times. That is, $n = q \cdot d + r$ where for the *remainder* r one has that $0 \leq r < d$. q is called the *quotient.*

Like 27 divided by 4 has remainder $r = 3$ and quotient $q = 6 : 27 = 6 \cdot 4 + 3$; while $-27 = (-7) \cdot 4 + 1$ shows that $q = -7, r = 1$.

Use the division algorithm in order to show that any list of *k* consecutive integers contains a member that is divisible by *k.* Use this in order to prove 0.16.

You may have to spend some time in order to figure out what went wrong in the proof that all horses are the same color.

For problem 0.18, you obviously have to establish that f_5 is divisible by 5.

Comments

Modern mathematics is built around the concept of a set. Basic set theoretic operations, like *union*, *intersection* and *complement* correspond to the logical connectives *or*, *and,* *negation*. *Containment* of sets corresponds to *implication*.

We may think that a set A is defined by a property; $p(a)$ means that the element a has property p :

$$A = \{a| \ p(a)\}$$

Because $p \wedge q$ has the same truth value as $q \wedge p$, namely T if and only if p and q are both true, we have that $A \cap B = B \cap A = \{a|p(a) \wedge q(a)\}$ where p defines A and q defines B. The logical connective *and* is certainly commutative. Intersection and union of sets is easily defined for any collection of sets, whether finite or infinite.

In mathematics everything can eventually be declared as a set. In particular the natural numbers:

$$0 = \emptyset, 1 = \{0\}, 2 = \{0,1\} \ldots$$

We take it for granted that there is a set ω of all *natural numbers*. This is called the *axiom of infinity*.

In somewhat more detail, Dedekind introduced induction the following way: The successor A^+ of any set A is the set $A \cup \{A\}$. For example $1 = 0^+, 2 = 1^+, \ldots$

A set is called inductive if it contains the empty set $\emptyset$ and with every element its successor.

The axiom of infinity says that there is an inductive set. The *set of natural numbers is then defined as the smallest inductive set*. The induction principle then is just a reformulation of this statement. Much more can be said about the definition of natural numbers, induction and the concept of infinite sets. But for this course we only need a working knowledge about induction and common sense intuition about natural numbers. For example, that every non-empty set of natural numbers contains a minimum.