Problems and Comments for Section 11, 12, 13

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(first edition)

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(second edition)

Comments: These three sections should be read simultaneously. It's best to start with section 12 on Homomorphisms. You may skip everything after example 4 on page 121.

A homomorphism is a map between similar algebras that preserves all the operations. Say, if $f = f^{\mathbf{A}}$ is some binary operation on the algebra $\mathbf{A}$ and $\varphi : \mathbf{A} \to \mathbf{B}$ a map from $\mathbf{A}$ to a similar algebra $\mathbf{B}$, where f is the operation $f^{\mathbf{B}}$, then φ is called a *homomorphism* if

$$\varphi(f^{\mathbf{A}}(a_1, a_2) = f^{\mathbf{B}}(\varphi(a_1), \varphi(a_1))$$

We may drop the superscripts and talk about the operation f on $\mathbf{A}$ and $\mathbf{B}$, respectively. A homomorphism between groups preserves the binary operation $*$, the unary operation $^{-1}$ and the constant e. Thus for a group homomorphism we stipulate the conditions

$$\varphi(a_1 * a_2) = \varphi(a_1) * \varphi(a_2)$$
$$\varphi(a^{-1}) = \varphi(a)^{-1}$$
$$\varphi(e) = e$$

As it is shown in the text (Theorem 12.4), a map between groups is already a homomorphism if the multiplication $*$ is preserved.

Recall that for any map f between sets A and B the equivalence relation $\ker(f)$ is defined on A by:

$$a_1 \sim a_2 \text{ iff } f(a_1) = f(a_2)$$

The set of equivalence classes is called the quotient set $C = A/\ker(f)$ and $q_f : A \to C, a \mapsto [a]$ is called the *quotient map*. Of course, q_f is surjective: Given any element $c \in C$, it is of the form $c = [a]$ for some $a \in A$. But then $q_f = [a]$.

Because f is constant on every class, we can define a map

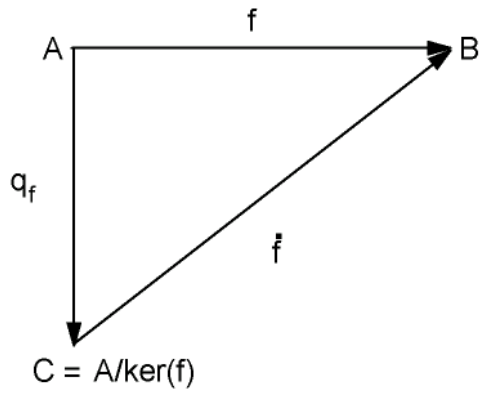
$$\dot{f} : C \to B, [a] \mapsto f(a)$$

If we take any element of C, then it is of the form $[a]$ for some $a \in A$. If a' is any other element in $[a]$ then by definition of the equivalence class of $[a]$ we have $f(a) = f(a')$. Thus, $\dot{f}$ is a *well defined* map.

We have that $\dot{f}$ is always injective, and bijective if and only if f is surjective. We have that:

$$\dot{f} \circ q_f = f$$

and this can be expressed by a *commutative diagram:*



It also says: Any map whatsoever is the composition of a surjective map, followed by an injective map.
For a homomorphism $f = \varphi$ and groups $\mathbf{A}$ and $\mathbf{B}$ one defines
$\ker(\varphi) = [e] = \{a|\varphi(a) = e\} = N_\varphi$ where N_φ is a *normal* subgroup of $\mathbf{A}$ and the equivalence classes turn out to be co-sets:

$$[a] = aN = Na$$

These cosets can be made to a group $\mathbf{C}$ by defining:

$$[a] * [b] = [a * b]$$
$$[a]^{-1} = [a^{-1}]$$
$$e = [e] = N$$

and with these definitions $\mathbf{C} = \mathbf{A}/N$ is a group and q_φ is (trivially) a homomorphism, and $\dot{\varphi}$ is a homomorphism. In particular, if φ is surjective then $\dot{\varphi}$ is an isomorphism. This is the **Fundamental Theorem on group homomorphisms** (**Theorem 13.2**).