Problems and Comments for Section 16

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Problems: 16.1, 16.6, 16.7, 16.12, 16.15, 16.16, 16.23, 16.24, 18.31, 18.32 (18.31 (b) requires the concept of characteristic. See below)

Comments: Most authors define rings as unitary, that is as rings with unit. This is a very easy section, most of the material is spruced up College Algebra. Some textbooks start for this reason with rings before going into groups.

The integers $\mathbb{Z}_n$ are prime examples of finite rings.

If n is not a prime then $n = k \cdot l$ for proper divisors k and l of n. Thus $[n] = [k] \cdot [l] = [0]$. Thus in this case, $\mathbb{Z}_n$ is not a domain. *Hence, if $\mathbb{Z}_n$ is a field then n must be prime.*

Now assume that n is prime and that $[n] = [k] \cdot [l] = [0]$. This means that $n|k \cdot l$. Because n is prime one has that $n|k$ or $n|l$. (This requires the prime factorization theorem for integers. Why?) Hence $[k] = 0$ or $[l] = 0$. Thus $\mathbb{Z}_n$ is a domain, hence, by 16.7, a field.

Theorem *$\mathbb{Z}_n$ is a field iff n is prime.*

Exercise 16.6 proves the rules for the algebra of fractions. Notice that the fractional calculus is not based on any order. The rules hold in any field. the two problems 18.31, 18.32 are also "college algebra" (pre calculus) and deal with order.

Let $\mathbf{A}$ be any ring with unit, call this unit 1. Similarly, define the sum of k-many 1 as the element k in the ring $\mathbf{A}$. Of course, $k = 0$ is possible, for example in $\mathbb{Z}_n$ one has that $n = [1] + [1] \cdots + [1] = 0$. A ring $\mathbf{A}$ has characteristic $k \in \mathbb{N}^+$ if k as an element of $\mathbf{A}$ is zero and is the smallest such element. Otherwise $\mathbf{A}$ has characteristic zero.