1 Problems and Comments For Section 2

Problems: 2.1, 2.5, 2.7, 2.8

Problem 2.13 requires some thought. It is optional.

A product term is defined recursively as follows:

1. \( p = a \) is a product where \( a \) is any element of the group \( G \).
2. If \( p_1 \) and \( p_2 \) are products, then \( p = (p_1 \cdot p_2) \) is a product.
3. All products are obtained that way.

For any two elements \( a, b \) one has that \( p = (a \cdot b) \) is a product. For three elements \( a, b, c \) there are two possibilities to form a product of these elements without changing the order: \((a \cdot (b \cdot c))\) and \(((a \cdot b) \cdot c)\). By associativity, both products have the same value.

The complexity of a product is the number of dots in it. \( p = a \) has complexity 0. If \( p \) has complexity \( n \) and \( q \) complexity \( m \) then \( (p \cdot q) \) has complexity \( n + m + 1 \). A product is a string of parenthesis (left and right), elements of \( G \) and dots. However such strings have to be constructed according to the rules 1. and 2. They have to be well-formed.

If, read from left to right, the elements in a product are \( a_1, a_2, \ldots, a_n \) then the product \( p = p(a_1, a_2, \ldots, a_n) \) has complexity \( n - 1 \). Given a list \( a_1, a_2, \ldots, a_n \) of elements, the normal product \( n(a_1, a_2, \ldots, a_n) \) of these elements is defined recursively by

1. \( n(a_n) = a_n \)
2. \( n(a_1, a_2, \ldots, a_n) = (a_1 \cdot n(a_2, \ldots, a_n)) \)

The claim of 2.13 now is

\[ p(a_1, \ldots, a_n) = n(a_1, \ldots, a_n) \]

for any product \( p(a_1, \ldots, a_n) \). Prove this by induction over the complexity of the product. 2.12 is a preparation for the general proof. Notice that for 2.12 and 2.13, only associativity is used.

**Question:** Why did I say “dots”, and not products, in the definition of complexity?

**Comments**

A group is most often defined as an algebraic system with three operations: \( \cdot,^{-1}, e \). Here \( \cdot \) is binary, \( -1 \) is unary and \( e \) is nullary. The axioms for a group then are all equations:

1. \( a \cdot (b \cdot c) = (a \cdot b) \cdot c \)
2. \( a \cdot e = e \cdot a = a \)
3. \( a \cdot a^{-1} = a^{-1} \cdot a = e \)
An algebraic system $\struct A = (A, \cdot)$ with only one binary associative operation is called a \textit{semi group}. A semi group with an identity (or unit) is called a \textit{monoid}. Thus a group is a monoid where every element has an inverse. Notice that under this definition, a group cannot be empty. It must at least have one element, the identity.

With this convention, the notation for the additive group of integers is $\struct Z = (\mathbb Z, +, -, 0)$. Here addition is the binary group operation. The multiplicative group of non-zero real numbers is $\struct R^* = (\mathbb R \setminus \{0\}, \cdot, -1, 1)$.

For any set $S$, $\Map(S) = \{ f | f : S \to S \}$ is the set of maps from $S$ to $S$. It is a semigroup. Here the operation is composition of maps. The identity map $\id : S \to S, x \mapsto x$ is the identity.

$$\Map(S) = (\{ f | f : S \to S \}, \circ, \id)$$

is the prototype of a monoid. It is not a group unless $S$ has only one element.