1 Problems and Comments For Section 2

Problems: 2.1, 2.5, 2.7, 2.8

Problem 2.13 requires some thought. It is optional.

A product term is defined *recursively* as follows:

1. $p = a$ is a product where a is any element of the group $\mathbf{G}$.
2. If $p_1$and p_2 are products, then $p = (p_1 \cdot p_2)$ is a product.
3. All products are obtained that way.

For any two elements a, b one has that $p = (a \cdot b)$ is a product. For three elements a, b, c there are two possibilities to form a product of these elements without changing the order: $((a \cdot (b \cdot c))$ and $((a \cdot b) \cdot c)$. By associativity, both products have the same value.

The *complexity* of a product is the number of dots in it. $p = a$ has complexity 0. If p has complexity n and q complexity m then $(p \cdot q)$ has complexity $n+m+1$. A product is a string of parenthesis (left and right), elements of $\mathbf{G}$ and dots. However such strings have to be constructed according to the rules 1. and 2. They have to be *well-formed.*

If, read from left to right, the elements in a product are $a_1, a_2, \ldots, a_n$ then the product $p = p(a_1, a_2, \ldots, a_n)$ has complexity $n - 1$.Given a list $a_1, a_2, \ldots, a_n$ of elements, the *normal product* $n(a_1, a_2, \ldots, a_n)$ of these elements is defined recursively by

1. $n(a_n) = a_n$
2. $n(a_1, a_2, \ldots, a_n) = (a_1 \cdot n(a_2, \ldots, a_n))$

The claim of 2.13 now is

$$p(a_1, \ldots, a_n) = n(a_1, \ldots, a_n)$$

for any product $p(a_1, \ldots, a_n)$.Prove this by induction over the complexity of the product. 2.12 is a preparation for the general proof. Notice that for 2.12 and 2.13, only associativity is used.

Question: Why did I say "dots", and not products, in the definition of complexity?

Comments

A group is most often defined as an algebraic system with three operations: $\cdot, ^{-1}, e$. Here $\cdot$ is binary, $^{-1}$ is unary and e is nullary. The axioms for a group then are all *equations*:

1. $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
2. $a \cdot e = e \cdot a = a$
3. $a \cdot a^{-1} = a^{-1} \cdot a = e$

An algebraic system $\mathbf{A} = (A, \cdot)$ with only one binary associative operation is called a *semi group.* A semi group with an identity (or unit) is called a *monoid.* Thus a group is a monoid where every element has an inverse. Notice that under this definition, a group cannot be empty. It must at least have one element, the identity.

With this convention, the notation for the additive group of integers is $\mathbb{Z} = (Z, +, -, 0)$. Here addition is the binary group operation. The multiplicative group of non-zero real numbers is $\mathbb{R}^* = (R \backslash \{0\}, \cdot, ^{-1}, 1)$.

For any set S, $Map(S) = \{f |\ f : S \to S\}$ is the set of maps f from S to S. It is a semigroup. Here the operation is composition of maps. The identity map $id : S \to S, x \mapsto x$ is the identity.

$$Map(S) = (\{f |\ f : S \to S\}, \circ, id)$$

is the prototype of a monoid. It is not a group unless S has only one element.