Problems and Comments For Section 3

Problems: 3.1, 3.4, 3.9, 3.10, 3.11, 3.12

Comments: For the multiplicative monoid of $n \times n$ –matrices one has that a left-inverse of a matrix A is automatically a right-inverse, thus an inverse.
This is proven in linear algebra and related to the

Theorem *The linear homogeneous system $AX = 0$ of n equations in n unknowns has only the trivial solution if and only if $AX = B$ has for every B (exactly) one solution. Here A is an $n \times n$ matrix and X and B are $n \times 1$ matrices.*

For the monoid of maps on a set X one has that the map $f : S \to S$ has a left inverse $g : S \to S,$ that is $g \circ f = id_S$ for some $g,$ if and only if f is *injective*, that is $f(x_1) = f(x_2)$ iff $x_1 = x_2$. And f has a right inverse $h : S \to S,$ that is $f \circ h = id_S$ for some h, if and only if f is surjective, that is for every $y \in S$ there is some x such that $f(x) = y$. It now follows::

Theorem *If a map $f : S \to S$ has a left as well a right inverse, then it has a unique inverse, which is the inverse map f^{-1} of f*

$$f(x) = y \Leftrightarrow f^{-1}(y) = x$$

For maps on *finite sets* S one has that $f : S \to S$ is injective if and only f is surjective. Why?

Exercise *Let $S = \mathbb{N}$ and $f : n \mapsto 2n$. Find a left inverse g and demonstrate that it is not a right inverse and that there are many left inverses for f. Now let $f : n \mapsto d(n)$, where $d(n)$ is the number of prime divisors of n. Find a right inverse g and demonstrate that it is not a left inverse and that there are many right inverses for f.*