Problems and Comments For Section 3

Problems: 3.1, 3.4, 3.9, 3.10, 3.11, 3.12

Comments: For the multiplicative monoid of $n \times n$ matrices one has that a left-inverse of a matrix $A$ is automatically a right-inverse, thus an inverse. This is proven in linear algebra and related to the

*Theorem* The linear homogeneous system $AX = 0$ of $n$ equations in $n$ unknowns has only the trivial solution if and only if $AX = B$ has for every $B$ (exactly) one solution. Here $A$ is an $n \times n$ matrix and $X$ and $B$ are $n \times 1$ matrices.

For the monoid of maps on a set $X$ one has that the map $f : S \to S$ has a left inverse $g : S \to S$, that is $g \circ f = \text{id}_S$ for some $g$, if and only if $f$ is injective, that is $f(x_1) = f(x_2)$ iff $x_1 = x_2$. And $f$ has a right inverse $h : S \to S$, that is $f \circ h = \text{id}_S$ for some $h$, if and only if $f$ is surjective, that is for every $y \in S$ there is some $x$ such that $f(x) = y$. It now follows:

*Theorem* If a map $f : S \to S$ has a left as well a right inverse, then it has a unique inverse, which is the inverse map $f^{-1}$ of $f$

$$f(x) = y \iff f^{-1}(y) = x$$

For maps on *finite sets* $S$ one has that $f : S \to S$ is injective if and only if $f$ is surjective. Why?

*Exercise* Let $S = \mathbb{N}$ and $f : n \mapsto 2n$. Find a left inverse $g$ and demonstrate that it is not a right inverse and that there are many left inverses for $f$. Now let $f : n \mapsto d(n)$, where $d(n)$ is the number of prime divisors of $n$. Find a right inverse $g$ and demonstrate that it is not a left inverse and that there are many right inverses for $f$. 