Problems and Comments for Section 5

Problems: 5.1, 5.2, 5.7, 5.18 (a), 5.19, 5.20, 5.21

Comments: For a given algebra $(A,(f_i)_{i\in I})$, a subset C is called closed if it is closed under all the operations: If for example f_i is a binary operation, say $\star$, and if $a,b \in C$ then also $a \star b \in C$. Also a closed subset must contain all constants, which are the values of the nullary operations.

For a group $\mathbf{G} = (G,\cdot,{}^{-1},e)$ this means that a subset C is closed if

1. If $a,b \in C$ then $a \cdot b \in C$.
2. If $a \in C$ then $a^{-1} \in C$.
3. $e \in C$.

One easily shows that the intersection of closed subsets is closed. The argument is the following. Assume that the $C_t, t \in T$, are closed subsets of the algebra A. Let:

$$C = \bigcap_{t\in T} C_t$$

We wish to show that C is closed. So assume, that $a,b \in C$. Then $a,b \in C_t$ for all $t \in T$. Because each C_t is closed we have that $a \star b \in C_t$ for all $t \in T$. Hence $a \star b \in C$. Of course, the whole algebra A is closed. The empty set $\emptyset$ is closed only if the algebra does not have constant operations.

It is clear that a closed subset of a group is a group where the group operations are restricted to the closed subset. This is so because the group axioms are equations. For example, associativity $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ holds for all group elements, in particular then for elements of the closed subset. Thus it is appropriate to call a closed subset of a group a subgroup. A closed subset of an arbitrary algebra is called a subalgebra.

For any subset S of an algebra A, the intersection of all closed subsets of A which contain S is a subalgebra. It is called the subalgebra generated by S and it is of course then the smallest subalgebra that contains S and is denoted $< S >$.

For a group G, and $S = \{a\}$, the subgroup generated by $\{a\}$ is then the cyclic group generated by a.