Problems and Comments for Section 7 and 9

Problems: 7.11, 7.12; 9.1, 9.6, 9.7, 9.8, 9.11, 9.12

Comments: The prototype of an equivalence relation on a set A is the *equivalence kernel* for a function $f : A \to B.$ Two elements in A are equivalent with respect to f if they have the same image under f :

$$a_1 \sim a_2 \Leftrightarrow f(a_1) = f(a_2)$$

For example, let S be the set of students in this class and let N be the set of legal first names. Then let $f : S \to N$ be the function which assigns to student s his first name. Then all students whose first name is Bob are equivalent, same for Mary etc. Students with the same first name then form an equivalence class.
Another familar example is given by a linear function

$$f : \mathbb{R}^3 \to \mathbb{R}, \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto a_1x_1 + a_2x_2 + a_3x_3 = (a_1, a_2, a_3) \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Here $\cdot$ stands for the product of the 1×3 matrix, a row, and the 3×1 matrix, a column. Any real number b defines an equivalence class which is the plane

$$a_1x_1 + a_2x_2 + a_3x_3 = b$$

Similarly, the function $f : \mathbb{R}^3 \to \mathbb{R}, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto r = \sqrt{x^2 + y^2 + z^2}$ defines an equivalence where two points P and Q of the space are equivalent if they have the same distance to the origin. Here the equivalence classes are the spheres with radii $r \geq 0$.
The following example is of great importance: Let $n > 0.$ For any integer $k \in \mathbb{Z},$ one has unique q_k and r_k with $0 \leq r_k < n,$ such that

$$k = q_k \cdot n + r_k$$

For $n = 2,$ and $k = 7$ one has that $7 = 3 \cdot 2 + 1.$ Thus $q_7 = 3$ and $r_7 = 1.$ Of course, $r_k = 0$ for even integers and $r_k = 1$ for the odd integers. Instead of q_k and r_k we should write $q_k(n)$ and $r_k(n)$ to indicate the dependency on n.
We now define two integers k, l as equivalent *modulo n* if $r_k(n) = r_l(n).$ For $n = 2$ we get two equivalence classes. The class of $k = 0,$ which is the set of even numbers, and the class of $k = 1,$ which is the set of odd numbers. We write

$$k \equiv l \text{ (mod } n)$$

If k and l are equivalent modulo n. We then have:

$$k \equiv l \text{ (mod } n) \text{ iff } r_k(n) = r_l(n) \text{ iff } n|(k-l) \text{ iff } (k-l) \in n\mathbb{Z}$$

The class $[0]_n$ of 0 is the subgroup $n\mathbb{Z}$ and the class $[k]_n$ of k is the right coset $n\mathbb{Z} + k$.