1. A sequence is Cauchy iff it is convergent. Every convergent sequence is bounded. □

2. $f$ is differentiable on $\mathbb{R}^n$ and $f'(x_0) = A$, $\forall x_0 \in \mathbb{R}^n$ since

$$\lim_{x \to x_0} \frac{\|f(x) - f(x_0) - A(x - x_0)\|}{\|x - x_0\|} = \lim_{x \to x_0} \frac{\|Ax - Ax_0 - A(x - x_0)\|}{\|x - x_0\|} = 0$$

3. (a) $|f(x, y) - f(0, 0)| = \left| \frac{x^3 - xy^2}{x^2 + y^2} - 0 \right| = |x| \frac{|x^2 - y^2|}{x^2 + y^2} \leq |x| \frac{x^2 + y^2}{x^2 + y^2} = |x|.$

By squeezing $f(x, y) \to f(0, 0)$ as $(x, y) \to (0, 0)$. By the Main Theorem About Continuity, $f$ is continuous at $(0, 0)$.

(b) By definition $\frac{\partial f}{\partial x}(0, 0) = \lim_{h \to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \to 0} \frac{h}{h} = 1.$

$\frac{\partial f}{\partial y}(0, 0) = \lim_{h \to 0} \frac{f(0, h) - f(0, 0)}{h} = \lim_{h \to 0} \frac{0}{h} = 0.$

(c) By contradiction. Suppose that $f$ is differentiable at $(0, 0)$. Then

$g(x, y) = \frac{|f(x, y) - f(0, 0) - f'(0, 0) \cdot (x - 0, y - 0)|}{\|(x - 0, y - 0)\|}$

must converge to 0 as $(x, y) \to (0, 0)$. Since $f(0, 0) = 0$ and $f'(0, 0) = \nabla f(0, 0) = (1, 0)$ by part (b), we have

$g(x, y) = \frac{|f(x, y) - 0 - (1, 0) \cdot (x, y)|}{\|(x, y)\|} = \frac{2|x|y^2}{(x^2 + y^2)^{3/2}}$

To show that the limit does not exist, consider the sequences $(1/k, 1/k) \to (0, 0)$ and $(1/k, 0) \to (0, 0)$ such that

$g(1/k, 1/k) = \frac{1}{\sqrt{2}}, \quad g(1/k, 0) = 0$

By the Main Theorem About Limits / Corollary 2 the limit does not exist, a contradiction. Hence, $f$ is not differentiable at $(0, 0)$. 1