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Existence and convergence of a discontinuous Galerkin method for the incompressible three-phase flow problem in porous media

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This paper presents and analyzes a discontinuous Galerkin method for the incompressible three-phase flow problem in porous media. We use a first-order time extrapolation, which allows us to solve the equations implicitly and sequentially. We show that the discrete problem is well posed, and obtain *a priori* error estimates. Our numerical results validate the theoretical results, i.e., the algorithm converges with first order.

Keywords: discontinuous Galerkin; three-phase flow; porous media; *a priori* error estimates.

1. Introduction

Subsurface modeling is important in improving the efficiency of clean-up strategies of contaminated subsurface or the long-term storage of carbon dioxide in subsurface. Incompressible systems of liquid phase, aqueous phase and vapor phase are mathematically modeled by nonlinear coupled partial differential equations that are challenging to analyze. This work formulates a numerical scheme for solving for the liquid pressure, the aqueous saturation and the vapor saturation using discontinuous Galerkin methods in space. This choice of primary unknowns is inspired from previous work by Shank & Vestal (1989), Hajibeygi & Tchelepi (2014), Cappanera & Riviere (2019a). The time marching uses a sequential and implicit time stepping. It allows us to avoid the use of iterative methods such as the L-scheme or Picard methods considered in, for example, Radu *et al.* (2018). Existence and uniqueness of the solutions is proved, and convergence of the numerical method is obtained by deriving *a priori* error estimates. These theoretical results are obtained under certain regularity assumptions on the data, such as boundedness and Lipschitz continuity. We refer to the reader to Alizadeh & Piri (2014) for a complete discussion on the advantages and limitations of such hypotheses. While the literature on computational modeling of three-phase flows is vast, to our knowledge there are no papers on the theoretical analysis of the discretization of the three-phase flow problem.

Ideal numerical methods for modeling multiphase flow in porous media are to be locally mass conservative to accurately track the propagation of the phases through the media. Heterogeneities of the porous media include highly discontinuous permeability fields with possibly local geological features like pinch-out. This implies that the numerical methods should handle discontinuous coefficients and unstructured grids. Discontinuous Galerkin (DG) methods are suitable methods thanks to their flexibility derived from the lack of a continuity constraint between approximations on neighboring cells. DG are known to be locally mass conservative, to handle highly varying permeability fields and to be accurate and robust on unstructured meshes. For these reasons, the literature on DG methods for porous media flows has exponentially increased over the past 20 years. The main drawback of these methods is their cost, which is higher than the cost of low-order finite difference methods and finite volume methods. DG has been applied to incompressible three-phase flow in Dong & Riviere (2016) and to compressible three-phase flow in Rankin & Riviere (2015), Cappanera & Riviere (2019b,c). In the absence of capillary pressure, DG is combined with the finite volume method in Natvig & Lie (2008), and with a mixed finite element method in Moortgat & Firoozabadi (2013, 2016). These papers show the convergence of the methods by performing numerical simulations on a sequence of uniformly refined meshes. The theoretical convergence of numerical methods for three-phase flows remains an open problem, and this paper provides the theoretical analysis of DG methods in the case of incompressible three-phase flows under certain conditions on the data. While the numerical analysis of three-phase flow is sparse, we note that the case of immiscible two-phase flows in porous media has been investigated in several papers. For instance for incompressible flows, finite difference methods have been analyzed in Douglas (1983), finite volume methods in Ohlberger (1997), Eymard *et al.* (2003), Michel (2003), DG methods in Epshteyn & Riviere (2009) and finite element methods in Chen & Ewing (2001), Girault *et al.* (2021a,b).

The paper is organized as follows. In Section 2 we present the problem considered and its mathematical formulation. Sections 3–4 describe the time and spatial discretization of our algorithm. Classical projection estimates and the hypothesis used for the numerical analysis of our method are detailed in Section 5. Then we show that the discrete problem is well posed in Section 6, and we establish *a priori* error estimates in Section 7. Eventually, we perform numerical investigations in Section 8 that recover the theoretical rate of convergence for various setups.

2. Problem description

Let p_j, s_j denote the pressure and the saturation, respectively, of the phase j, where $j = \ell, v, a$ (liquid, vapor and aqueous). The saturation for phase j at a point $\boldsymbol{x}$ in the domain $\Omega \subset \mathbb{R}^d$, with $d = 2, 3$, is defined as the ratio of the volume of phase j to the total pore volume in a representative elementary volume centered around the point $\boldsymbol{x}$. Thus, the saturations satisfy

$$s_\ell + s_v + s_a = 1. \tag{2.1}$$

Assuming that the phase densities and the porosity are constant, the mass conservation equation of each component is expressed as

$$\phi \partial_t s_j - \nabla \cdot \Big(\kappa \lambda_j \Big(\nabla p_j - \rho_j \boldsymbol{g}\Big)\Big) = q_j, \quad j = \ell, a, v, \tag{2.2}$$

where κ is the absolute permeability, ρ_j denotes the density of phase j, λ_j denotes the mobility of phase j and ϕ is the porosity of the medium. The mobility λ_j is defined as $\lambda_j = k_{rj}/\mu_j$, where k_{rj} and μ_j represent the relative permeability and viscosity of phase j, respectively. Gravity is denoted by $\boldsymbol{g}$ and q_ℓ, q_v and q_a are source/sink terms. The differences between phase pressures are capillary pressures $p_{c,v}$

and $p_{c,a}$ defined as

$$p_{c,v} = p_v - p_\ell, \quad p_{c,a} = p_\ell - p_a. \tag{2.3}$$

From the set of unknowns (saturations and pressures), we choose for primary unknowns the liquid pressure p_ℓ, the aqueous saturation s_a and the vapor saturation s_v. For clarity, we explicitly write the dependence of the different quantities with respect to the primary unknowns:

$$p_{c,v}(s_v), \quad p_{c,a}(s_a), \quad \lambda_\ell(s_v, s_a), \quad \lambda_v(s_v, s_a), \quad \lambda_a(s_v, s_a), \tag{2.4}$$

$$\mu_\ell(p_\ell), \quad \mu_v(s_v, s_a), \quad \mu_a(s_v, s_a). \tag{2.5}$$

Moreover, the capillary pressures are assumed to be differentiable, $\partial_{s_a} p_{c,a}$ is a negative function and $\partial_{s_v} p_{c,v}$ is a positive function.

2.1 *Rewritten equations*

Summing the three mass conservation equations (2.2) and using the definition of the capillary pressure (2.3) yields the liquid pressure equation

$$-\nabla \cdot \left(\lambda_t \kappa \nabla p_\ell\right) - \nabla \cdot \left(\lambda_v \kappa \nabla p_{c,v}\right) + \nabla \cdot \left(\lambda_a \kappa \nabla p_{c,a}\right) = q_t - \nabla \cdot \left(\kappa \left(\rho\lambda\right)_t \boldsymbol{g}\right), \tag{2.6}$$

where

$$\left(\rho\lambda\right)_t = \rho_\ell \lambda_\ell + \rho_v \lambda_v + \rho_a \lambda_a, \quad \lambda_t = \lambda_\ell + \lambda_v + \lambda_a, \quad q_t = q_\ell + q_v + q_a. \tag{2.7}$$

Using the capillary pressure $p_{c,a}$, the mass conservation (2.2) satisfied by the aqueous saturation can be rewritten

$$\phi \partial_t s_a + \nabla \cdot \left(\kappa \lambda_a \partial_{s_a} p_{c,a} \nabla s_a\right) - \nabla \cdot \left(\kappa \lambda_a \nabla p_\ell\right) = q_a - \nabla \cdot \left(\rho_a \kappa \lambda_a \boldsymbol{g}\right). \tag{2.8}$$

Similarly, the vapor saturation s_v satisfies the following equation, derived from (2.2) with $j = v$:

$$\phi \partial_t s_v - \nabla \cdot \left(\kappa \lambda_v \partial_{s_v} p_{c,v} \nabla s_v\right) - \nabla \cdot \left(\kappa \lambda_v \nabla p_\ell\right) = q_v - \nabla \cdot \left(\rho_v \kappa \lambda_v \boldsymbol{g}\right). \tag{2.9}$$

These equations are complemented with Dirichlet and Neumann boundary conditions. The boundary of the computational domain Ω is decomposed as

$$\partial\Omega = \Gamma_{\mathrm{D}}^{p_\ell} \cup \Gamma_{\mathrm{N}}^{p_\ell} = \Gamma_{\mathrm{D}}^{s_a} \cup \Gamma_{\mathrm{N}}^{s_a} = \Gamma_{\mathrm{D}}^{s_v} \cup \Gamma_{\mathrm{N}}^{s_v}, \tag{2.10}$$

with $|\Gamma_{\mathrm{D}}^{p_\ell}| > 0, |\Gamma_{\mathrm{D}}^{s_a}| > 0, |\Gamma_{\mathrm{D}}^{s_v}| > 0$. The Dirichlet boundary conditions imposed on $\Gamma_{\mathrm{D}}^{p_\ell}$, $\Gamma_{\mathrm{D}}^{s_a}$ and $\Gamma_{\mathrm{D}}^{s_v}$ are denoted by p_ℓ^{bdy}, s_a^{bdy}, s_v^{bdy}. The Neumann boundary conditions imposed on $\Gamma_{\mathrm{N}}^{p_\ell}$, $\Gamma_{\mathrm{N}}^{s_a}$ and $\Gamma_{\mathrm{N}}^{s_v}$ are given by

$$\left(\lambda_t \kappa \nabla p_\ell + \lambda_v \kappa \nabla p_{c,v} - \lambda_a \kappa \nabla p_{c,a} - \kappa (\rho\lambda)_t \boldsymbol{g}\right) \cdot \boldsymbol{n} = j_p^{\mathrm{N}}, \tag{2.11a}$$

$$\left(-\kappa\lambda_a\partial_{s_a}p_{c,a}\nabla s_a + \kappa\lambda_a\nabla p_\ell - \rho_a\kappa\lambda_a\boldsymbol{g}\right)\cdot\boldsymbol{n} = j_{s_a}^{\mathrm{N}}, \tag{2.11b}$$

$$\left(\kappa\lambda_v\partial_{s_v}p_{c,v}\nabla s_v + \kappa\lambda_v\nabla p_\ell - \rho_v\kappa\lambda_v\boldsymbol{g}\right)\cdot\boldsymbol{n} = j_{s_v}^{\mathrm{N}}, \tag{2.11c}$$

where $\boldsymbol{n}$ represents the outward unit normal vector to the boundary $\partial\Omega$.

3. Time discretization

For the time discretization, we use a backward Euler method and partition the time interval $[0, T]$ using a time step $\tau > 0$ such that $N\tau = T$. In the rest of the paper, we define $t_n = n\tau$ for any integer $0 \leqslant n \leqslant N$, and for any time-dependent function f, we define $f^n = f|_{t=t_n}$.

3.1 *Liquid pressure*

The time discretization of the liquid pressure (2.6) reads

$$-\nabla\cdot\left(\lambda_t^n\kappa\nabla p_\ell^{n+1}\right) = q_t^{n+1} - \nabla\cdot\left(\kappa(\rho\lambda)_t^n\boldsymbol{g}\right) + \nabla\cdot\left(\lambda_v^n\kappa\nabla p_{c,v}^n\right) - \nabla\cdot\left(\lambda_a^n\kappa\nabla p_{c,a}^n\right). \tag{3.1}$$

3.2 *Aqueous saturation*

The time discretization of the aqueous saturation equation (2.8) is

$$\phi\frac{s_a^{n+1}-s_a^n}{\tau} + \nabla\cdot\left(\kappa\lambda_a^n\left(\partial_{s_a}p_{c,a}\right)^n\nabla s_a^{n+1}\right) = q_a^{n+1} + \nabla\cdot\left(\kappa\lambda_a^n\left(\nabla p_\ell^{n+1} - \rho_a\boldsymbol{g}\right)\right). \tag{3.2}$$

Note that $\partial_{s_a}p_{c,a}$ is negative. Therefore, with $\left(\partial_{s_a}p_{c,a}\right)^{n,+} = -\left(\partial_{s_a}p_{c,a}\right)^n$, we may write (3.2) as

$$\phi\frac{s_a^{n+1}-s_a^n}{\tau} - \nabla\cdot\left(\kappa\lambda_a^n\left(\partial_{s_a}p_{c,a}\right)^{n,+}\nabla s_a^{n+1}\right) = q_a^{n+1} + \nabla\cdot\left(\kappa\lambda_a^n\left(\nabla p_\ell^{n+1} - \rho_a\boldsymbol{g}\right)\right). \tag{3.3}$$

3.3 *Vapor saturation*

The time discretization of the vapor saturation equation (2.9) reads

$$\phi\frac{s_v^{n+1}-s_v^n}{\tau} - \nabla\cdot\left(\kappa\lambda_v^n\left(\partial_{s_v}p_{c,v}\right)^n\nabla s_v^{n+1}\right) = q_v^{n+1} + \nabla\cdot\left(\kappa\lambda_v^n\left(\nabla p_\ell^{n+1} - \rho_v\boldsymbol{g}\right)\right). \tag{3.4}$$

4. Spatial discretization

For the spatial discretization, we use an interior penalty discontinuous Galerkin method. The domain Ω is discretized with a conforming, shape-regular mesh $\mathscr{E}_h$ consisting of simplices or quadrilateral and hexaedral elements. We denote by h_e and h_K the size of an edge (or face for $d = 3$) e and an element K, respectively. Moreover, we define the mesh size $h = \max_{K\in\mathscr{E}_h} h_K$. For any quadrilateral element K, we define the two-dimensional local polynomial space $\mathbb{P}_{k_1,k_2}(K)$ as

$$\mathbb{P}_{k_1,k_2}(K) = \left\{ p(x,y) \,\middle|\, p(x,y) = \sum_{i\leqslant k_1}\sum_{j\leqslant k_2} a_{ij}x^i y^j \right\}. \tag{4.1}$$

The three-dimensional local polynomial space $\mathbb{P}_{k_1,k_2,k_3}(K)$ is defined similarly. Finally, we define $\mathbb{Q}_k(K) = \mathbb{P}_{k,k}(K)$ for $d = 2$, and $\mathbb{Q}_k(K) = \mathbb{P}_{k,k,k}(K)$ for $d = 3$. The space of discontinuous piecewise linear polynomials is denoted by X_h. If $\mathscr{E}_h$ consists of quadrilateral or hexahedral elements, the space X_h is defined by

$$X_h = \left\{ v \in L^2(\Omega) : v|_K \in \mathbb{Q}_1(K) \forall K \in \mathscr{E}_h \right\}. \tag{4.2}$$

The discrete liquid pressure, aqueous saturation and vapor saturation at time t_n are denoted by $P_h^n, S_{a_h}^n$ and $S_{v_h}^n$ respectively; they belong to the finite-dimensional spaces X_h. The Dirichlet boundary conditions are imposed strongly; thus, we assume that the data $p_\ell^{\mathrm{bdy}}, s_a^{\mathrm{bdy}}, s_v^{\mathrm{bdy}}$ are traces of functions in X_h. This assumption is in agreement with realistic simulations where the Dirichlet data are simply constants on the Dirichlet boundaries. We will make use of the following finite-dimensional spaces for the test functions:

$$X_{h,\Gamma_{\mathrm{D}}^{p_\ell}} = X_h \cap \{v = 0 \text{ on } \Gamma_{\mathrm{D}}^{p_\ell}\}, \quad X_{h,\Gamma_{\mathrm{D}}^{s_a}} = X_h \cap \{v = 0 \text{ on } \Gamma_{\mathrm{D}}^{s_a}\}, \quad X_{h,\Gamma_{\mathrm{D}}^{s_v}} = X_h \cap \{v = 0 \text{ on } \Gamma_{\mathrm{D}}^{s_v}\}. \tag{4.3}$$

We also define the Raviart–Thomas space $\mathbb{RT}_0$:

$$\mathbb{RT}_0 = \left\{ \boldsymbol{u} \in H(\mathrm{div}, \Omega) : \boldsymbol{u}|_K \in \mathbb{RT}_0(K) \forall K \in \mathscr{E}_h \right\}, \tag{4.4}$$

where

$$\mathbb{RT}_0(K) = \begin{cases} \mathbb{P}_{1,0}(K) \times \mathbb{P}_{0,1}(K), & d = 2, \\ \mathbb{P}_{1,0,0}(K) \times \mathbb{P}_{0,1,0}(K) \times \mathbb{P}_{0,0,1}(K), & d = 3. \end{cases} \tag{4.5}$$

We note that the above spaces can be defined similarly if one uses simplex elements. The set of interior faces is denoted by Γ_h. For any interior face e, we fix a unit normal vector $\boldsymbol{n}_e$, and we denote by K_1 and K_2 the elements that share the face e such that $\boldsymbol{n}_e$ points from K_1 into K_2. For any function $f \in X_h$, we define the jump operator $[\cdot]$ on interior faces as $[f] = f_1 - f_2$, where $f_i = f|_{K_i}$. Moreover, we define the weighted average operator $\{\cdot\}$ on interior faces as $\{A\nabla f \cdot \boldsymbol{n}_e\} = \omega_1 A_1 \nabla f_1 \cdot \boldsymbol{n}_e + \omega_2 A_2 \nabla f_2 \cdot \boldsymbol{n}_e$, where $\omega_1 = A_2(A_1 + A_2)^{-1}$ and $\omega_2 = A_1(A_1 + A_2)^{-1}$. Note that the standard average operator with weights $\omega_1 = \omega_2 = 1/2$ is denoted by $\{\cdot\}_{\frac{1}{2}}$. On boundary faces, the jump and weighted average operators are defined as $[f] = \{f\} = f$. In the following, the L^2 inner product over Ω is denoted by $(\cdot,\cdot)$. The parameters $\theta_{p_\ell}, \theta_{s_a}, \theta_{s_v}$ take values -1, 0, 1, which respectively correspond to symmetric, incomplete and nonsymmetric interior penalty discontinuous Galerkin.

4.1 *Liquid pressure*

The discrete problem for the liquid pressure reads as follows: find $P_h^{n+1} \in X_h$ such that $P_h^{n+1} = p_\ell^{\mathrm{bdy}}$ on $\Gamma_{\mathrm{D}}^{p_\ell}$ and the following relation is satisfied for all $w_h \in X_{h,\Gamma_{\mathrm{D}}^{p_\ell}}$:

$$b_p^n(P_h^{n+1}, w_h) = f_p^n(w_h), \tag{4.6}$$

where $b_p^n(P_h^{n+1}, w_h) = b_p(P_h^{n+1}, w_h; P_h^n, S_{a_h}^n, S_{v_h}^n), f_p^n(w_h) = f_p(w_h; P_h^n, S_{a_h}^n, S_{v_h}^n)$, with b_p and f_p defined as

$$\begin{aligned} b_p(v_h, w_h; P_h^n, S_{a_h}^n, S_{v_h}^n) = &\sum_{K\in\mathscr{E}_h} \int_K \lambda_t^n \kappa \nabla v_h \cdot \nabla w_h + \sum_{e\in\Gamma_h} \alpha_{p_\ell,e} h_e^{-1} \int_e \eta_{p_\ell,e}^n \left[v_h\right]\left[w_h\right] \\ &- \sum_{e\in\Gamma_h} \int_e \left\{\lambda_t^n \kappa \nabla v_h \cdot \boldsymbol{n}_e\right\}\left[w_h\right] + \theta_{p_\ell} \sum_{e\in\Gamma_h} \int_e \left\{\lambda_t^n \kappa \nabla w_h \cdot \boldsymbol{n}_e\right\}\left[v_h\right] \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} f_p(w_h; P_h^n, S_{a_h}^n, S_{v_h}^n) = &(q_t^{n+1}, w_h) + \sum_{e\in\Gamma_{\mathrm{N}}^{p_\ell}} \int_e j_p^{\mathrm{N}} w_h \\ &- \sum_{K\in\mathscr{E}_h} \int_K \left(\lambda_v^n \kappa \nabla p_{c,v}^n - \lambda_a^n \kappa \nabla p_{c,a}^n - \kappa \left(\rho\lambda\right)_t^n \boldsymbol{g}\right) \cdot \nabla w_h \\ &+ \sum_{e\in\Gamma_h} \int_e \left\{\lambda_v^n \kappa \nabla p_{c,v}^n \cdot \boldsymbol{n}_e\right\}\left[w_h\right] - \sum_{e\in\Gamma_h} \int_e \left\{\lambda_a^n \kappa \nabla p_{c,a}^n \cdot \boldsymbol{n}_e\right\}\left[w_h\right] \\ &- \sum_{e\in\Gamma_h} \int_e \left\{\kappa \left(\rho\lambda\right)_t^n \boldsymbol{g} \cdot \boldsymbol{n}_e\right\}\left[w_h\right]. \end{aligned} \tag{4.8}$$

We recall that λ_t^n, $(\rho\lambda)_t^n$, λ_i^n for $i = v, \ell, a$ are the functions λ_t, $(\rho\lambda)_t$, λ_i evaluated at the discrete solutions (discrete pressures and saturations) at time t_n. The penalty parameter $\alpha_{p_\ell,e}$ is a positive constant such that $0 < \alpha_{p_\ell,*} \leqslant \alpha_{p_\ell,e} \leqslant \alpha_{p_\ell}^*$, and the penalty parameter $\eta_{p_\ell,e}^n$ depends on the absolute permeability and mobilities in the following way:

$$\eta_{p_\ell,e}^n = \mathscr{H}\left(\left(\kappa\lambda_t^n\right)|_{K_1}, \left(\kappa\lambda_t^n\right)|_{K_2}\right) \quad \forall\, e = \partial K_1 \cap \partial K_2, \tag{4.9}$$

where $\mathscr{H}$ is the harmonic average function:

$$\mathscr{H}(x_1, x_2) = \frac{2x_1x_2}{x_1 + x_2}. \tag{4.10}$$

4.2 *Aqueous saturation*

The discrete problem for the aqueous saturation reads as follows: find $S_{a_h}^{n+1} \in X_h$ such that $S_{a_h}^{n+1} = s_a^{\mathrm{bdy}}$ on $\Gamma_{\mathrm{D}}^{s_a}$ and such that the following relation is satisfied for all $w_h \in X_{h,\Gamma_{\mathrm{D}}^{s_a}}$:

$$\frac{1}{\tau}(\phi S_{a_h}^{n+1}, w_h) + b_a^n(S_{a_h}^{n+1}, w_h) = \frac{1}{\tau}(\phi S_{a_h}^{n}, w_h) + f_a^n(w_h), \tag{4.11}$$

where $b_a^n(S_{a_h}^{n+1}, w_h) = b_a(S_{a_h}^{n+1}, w_h; P_h^{n+1}, S_{a_h}^n, S_{v_h}^n)$, $f_a^n(w_h) = f_a(w_h; P_h^{n+1}, S_{a_h}^n, S_{v_h}^n)$, with b_a and f_a defined as

$$\begin{aligned} b_a(v_h, w_h; P_h^{n+1}, S_{a_h}^n, S_{v_h}^n) &= \sum_{K\in\mathscr{E}_h} \int_K \kappa\lambda_a^n \Big(\partial_{s_a} p_{c,a}\Big)^{+,n} \nabla v_h \cdot \nabla w_h \\ &- \sum_{e\in\Gamma_h} \int_e \Big\{\kappa\lambda_a^n \Big(\partial_{s_a} p_{c,a}\Big)^{+,n} \nabla v_h \cdot \boldsymbol{n}_e\Big\}\Big[w_h\Big] + \sum_{e\in\Gamma_h} \alpha_{s_a,e} h_e^{-1} \int_e \eta_{s_a,e}^n \Big[v_h\Big]\Big[w_h\Big] \\ &+ \theta_{s_a} \sum_{e\in\Gamma_h} \int_e \Big\{\kappa\lambda_a^n \Big(\partial_{s_a} p_{c,a}\Big)^{+,n} \nabla w_h \cdot \boldsymbol{n}_e\Big\}\Big[v_h\Big] \end{aligned} \tag{4.12}$$

and

$$\begin{aligned} f_a(w_h; P_h^{n+1}, S_{a_h}^n, S_{v_h}^n) &= (q_a^{n+1}, w_h) + \sum_{K\in\mathscr{E}_h} \int_K \Big(\lambda_a^n \boldsymbol{u}_h^{n+1} + \kappa\rho_a\lambda_a^n \boldsymbol{g}\Big) \cdot \nabla w_h + \sum_{e\in\Gamma_{\mathrm{N}}^{s_a}} \int_e j_{s_a}^{\mathrm{N}} w_h \\ &- \sum_{e\in\Gamma_h} \int_e \big(\lambda_a^n\big)_{s_a}^{\uparrow} \boldsymbol{u}_h^{n+1} \cdot \boldsymbol{n}_e \Big[w_h\Big] - \sum_{e\in\Gamma_h} \int_e \Big\{\rho_a\kappa\lambda_a^n \boldsymbol{g} \cdot \boldsymbol{n}_e\Big\}\Big[w_h\Big]. \end{aligned} \tag{4.13}$$

In (4.13), the vector $\boldsymbol{u}_h^{n+1}$ is the projection of the approximation of the Darcy velocity onto the Raviart–Thomas space $\mathbb{RT}_0$ (see the exact definition of operator Π_{RT} in Section 4.5):

$$\boldsymbol{u}_h^{n+1} = \Pi_{\mathrm{RT}}(-\kappa\nabla P_h^{n+1}).$$

The upwind operator $(\cdot)_{s_a}^{\uparrow}$ is defined as follows. For readability, let $D = \lambda_a^n$ and $D^g = \rho_a\kappa\lambda_a^n$. For an interior edge e shared by two elements K_1 and K_2, we have

$$(D)_{s_a}^{\uparrow} = \begin{cases} D|_{K_1} & \text{if } \{D\boldsymbol{u}_h^{n+1} + D^g \boldsymbol{g}\}_{\frac{1}{2}} \cdot \boldsymbol{n}_e \geqslant 0, \\ D|_{K_2} & \text{otherwise.} \end{cases} \tag{4.14}$$

The penalty parameter $\alpha_{s_a,e}$ is a positive constant such that $0 < \alpha_{s_a,*} \leqslant \alpha_{s_a,e} \leqslant \alpha_{s_a}^*$, and the parameter $\eta_{s_a,e}^n$ is defined on the interior faces by

$$\eta_{s_a,e}^n = \mathscr{H}\Big(\Big(\kappa(\partial_{s_a} p_{c,a})^{+,n}\lambda_a^n\Big)|_{K_1}, \Big(\kappa(\partial_{s_a} p_{c,a})^{+,n}\lambda_a^n\Big)|_{K_2}\Big) \quad \forall\, e = \partial K_1 \cap \partial K_2. \tag{4.15}$$

4.3 *Vapor saturation*

The discrete problem for the vapor saturation reads as follows: find $S_{v_h}^{n+1} \in X_h$ such that $S_{v_h}^{n+1} = s_v^{\mathrm{bdy}}$ on $\Gamma_{\mathrm{D}}^{s_v}$ and such that the following relation is satisfied for all $w_h \in X_{h,\Gamma_{\mathrm{D}}^{s_v}}$:

$$\frac{1}{\tau}(\phi S_{v_h}^{n+1}, w_h) + b_v^n(S_{v_h}^{n+1}, w_h) = \frac{1}{\tau}(\phi S_{v_h}^n, w_h) + f_v^n(w_h), \tag{4.16}$$

where $b_v^n(S_{v_h}^{n+1}, w_h) = b_v(S_{v_h}^{n+1}, w_h; P_h^{n+1}, S_{a_h}^{n+1}, S_{v_h}^n)$, $f_v^n(w_h) = f_v(w_h; P_h^{n+1}, S_{a_h}^{n+1}, S_{v_h}^n)$, with b_v and f_v defined as

$$\begin{aligned} b_v(v_h, w_h; P_h^{n+1}, S_{a_h}^{n+1}, S_{v_h}^n) &= \sum_{K\in\mathscr{E}_h} \int_K \kappa \lambda_v^n \partial_{s_v} p_{c,v}^n \nabla v_h \cdot \nabla w_h \\ &- \sum_{e\in\Gamma_h} \int_e \left\{ \kappa \lambda_v^n \partial_{s_v} p_{c,v}^n \nabla v_h \cdot \boldsymbol{n}_e \right\} \left[w_h \right] + \sum_{e\in\Gamma_h} \alpha_{s_v,e} h_e^{-1} \int_e \eta_{s_v,e}^n \left[v_h \right] \left[w_h \right] \\ &+ \theta_{s_v} \sum_{e\in\Gamma_h} \int_e \left\{ \kappa \lambda_v^n \partial_{s_v} p_{c,v}^n \nabla w_h \cdot \boldsymbol{n}_e \right\} \left[v_h \right] \end{aligned} \tag{4.17}$$

and

$$\begin{aligned} f_v(w_h; P_h^{n+1}, S_{a_h}^{n+1}, S_{v_h}^n) &= (q_v^{n+1}, w_h) + \sum_{K\in\mathscr{E}_h} \int_K \left(\lambda_v^n \boldsymbol{u}_h^{n+1} + \kappa \rho_v \lambda_v^n \boldsymbol{g} \right) \cdot \nabla w_h \\ &+ \sum_{e\in\Gamma_{\mathrm{N}}^{s_v}} \int_e j_{s_v}^{\mathrm{N}} w_h - \sum_{e\in\Gamma_h} \int_e \left(\lambda_v^n \right)_{s_v}^{\uparrow} \boldsymbol{u}_h^{n+1} \cdot \boldsymbol{n}_e \left[w_h \right] - \sum_{e\in\Gamma_h} \int_e \left\{ \rho_v \kappa \lambda_v^n \boldsymbol{g} \cdot \boldsymbol{n}_e \right\} \left[w_h \right], \end{aligned} \tag{4.18}$$

where $(\cdot)_{s_v}^{\uparrow}$ denotes the upwind average operator that is defined similarly to $(\cdot)_{s_a}^{\uparrow}$, but with $D = \lambda_v^n$ and $D^g = \rho_v \kappa \lambda_v^n$. The penalty parameter $\alpha_{s_v,e}$ is a positive constant such that $0 < \alpha_{s_v,*} \leqslant \alpha_{s_v,e} \leqslant \alpha_{s_v}^*$, and $\eta_{s_v,e}^n$ is defined by

$$\eta_{s_v,e}^n = \mathscr{H}\Big(\big(\kappa (\partial_{s_v} p_{c,v})^n \lambda_v^n\big)|_{K_1}, \big(\kappa (\partial_{s_v} p_{c,v})^n \lambda_v^n\big)|_{K_1} \Big). \tag{4.19}$$

4.4 *Starting the algorithm*

To start the algorithms, we choose the L^2 projections of the unknowns at time t_0. Let Π_h be the L^2 projection onto X_h:

$$P_h^0 = \Pi_h p_\ell^0, \quad S_{a_h}^0 = \Pi_h s_a^0, \quad S_{v_h}^0 = \Pi_h s_v^0, \tag{4.20}$$

where p_ℓ^0, s_a^0, s_v^0 are the exact solutions at time t_0.

4.5 *Raviart–Thomas projection*

The Raviart–Thomas projection, $\boldsymbol{u}_h^{n+1} = \Pi_{\mathrm{RT}}(-\kappa \nabla P_h^{n+1})$, is defined by the following equations:

$$\int_e \boldsymbol{u}_h^{n+1} \cdot \boldsymbol{n}_e q_h = -\int_e \{\kappa \nabla P_h^{n+1} \cdot \boldsymbol{n}_e\} q_h + \alpha_{p_\ell,e} h_e^{-1} \int_e \eta_{p_\ell,e}^n [P_h^{n+1}] q_h \quad \forall\, q_h \in \mathbb{Q}_0(e), \forall\, e \in \Gamma_h, \tag{4.21a}$$

$$\int_e \boldsymbol{u}_h^{n+1} \cdot \boldsymbol{n}_e q_h = -\int_e \kappa \nabla P_h^{n+1} \cdot \boldsymbol{n}_e q_h \qquad \forall\, q_h \in \mathbb{Q}_0(e), \forall\, e \in \partial\Omega. \tag{4.21b}$$

This projection was introduced for elliptic partial differential equations in Ern *et al.* (2007) for spaces of the same order; we apply it here to Raviart–Thomas spaces with a degree less than the DG spaces.

5. Preliminaries

In this section we establish some notation and recall some well-known results from finite element analysis that will be used in the rest of the paper. Finally, we list the hypotheses assumed in this work.

5.1 *Notation and useful results*

The L^2 norm over a set D is denoted by $\|\cdot\|_{L^2(D)}$. When $D = \Omega$, the subscript will be omitted. Let us define the space $X(h) = X_h + H^2(\Omega)$. For functions $w \in X(h)$, we define the broken gradient $\nabla_h w$ by $(\nabla_h w)|_K = \nabla(w|_K)$. The space $X(h)$ is endowed with the *coercivity* norm for all $w \in X(h)$:

$$|||w||| := \left(\|\nabla_h w\|^2 + |w|_{\mathrm{J}}^2\right)^{1/2}, \quad |w|_{\mathrm{J}} = \left(\sum_{e\in\Gamma_h} h_e^{-1} \|[w]\|_{L^2(e)}^2\right)^{1/2}. \tag{5.1}$$

Additionally, we introduce the following norm on $X(h)$:

$$|||w|||_* := \left(|||w|||^2 + \sum_{K\in\mathscr{E}_h} h_K \|\nabla w|_K \cdot \boldsymbol{n}_K\|_{L^2(\partial K)}^2\right)^{1/2}. \tag{5.2}$$

The following classical finite element results will be used in the analysis carried out in Sections 6 and 7.

LEMMA 5.1 (Trace inequality). Let $\mathscr{E}_h$ be a shape-regular mesh with parameter C_{shape}. Then, for all $w_h \in X_h$, all $K \in \mathscr{E}_h$ and all $e \in \partial K$, we have

$$\|w_h\|_{L^2(e)} \leqslant C_{\mathrm{tr}} h_K^{-1/2} \|w_h\|_{L^2(K)}, \tag{5.3}$$

where $C_{\mathrm{tr}} > 0$ depends only on C_{shape}.

LEMMA 5.2 (Discrete Poincaré inequality; Brenner, 2003). For all w in the broken Sobolev space $H^1(\mathscr{E}_h)$, there exists a constant $C_P > 0$ independent of h such that

$$\|w\| \leqslant C_P |||w|||. \tag{5.4}$$

We denote by $\pi_{h,\Gamma}$ the L^2-orthogonal projection onto $X_{h,\Gamma}$ for $\Gamma \in \{\Gamma_{\mathrm{D}}^{p_\ell}, \Gamma_{\mathrm{D}}^{s_a}, \Gamma_{\mathrm{D}}^{s_v}\}$. The following lemma recalls approximation estimates that are later used in the analysis of the numerical scheme introduced in Section 4.

LEMMA 5.3 (L^2-orthogonal projection approximation bounds). For any element $K \in \mathscr{E}_h$, for all $s \in^i \{0, 1, 2\}$ and all $w \in H^s(K)$, there holds

$$|w - \pi_{h,\Gamma} w|_{H^m(K)} \leqslant C h_K^{s-m} |w|_{H^s(K)} \quad \forall\, m \in \Big\{0, \dots, s\Big\}, \tag{5.5}$$

where C is independent of both K and h_K. Moreover, if $s \geqslant 1$, then for all $K \in \mathscr{E}_h$ and all $e \in \partial K$, there holds

$$\|w - \pi_{h,\Gamma} w\|_{L^2(e)} \leqslant C h_K^{s-1/2} |w|_{H^s(K)}, \tag{5.6}$$

and if $s \geqslant 2$,

$$\big\|\nabla\big(w - \pi_{h,\Gamma} w\big)\big|_K \cdot \boldsymbol{n}_K\big\|_{L^2(e)} \leqslant C h_K^{s-3/2} |w|_{H^s(K)}. \tag{5.7}$$

Note that these results imply that

$$|||w - \pi_{h,\Gamma} w|||_* \leqslant C h_K^{s-1} |w|_{H^s(\Omega)}. \tag{5.8}$$

The projected velocity $\boldsymbol{u}_h^{n+1}$ defined by (4.21a)–(4.21b) satisfies the following approximation bound.

LEMMA 5.4 Assume p_ℓ belongs to $L^2(0, T; H^2(\Omega))$. There is a positive constant independent of h and τ such that

$$\|\boldsymbol{u}_h^{n+1} + \kappa \nabla_h P_h^{n+1}\| \leqslant C|||P_h^{n+1} - p_\ell^{n+1}||| + Ch. \tag{5.9}$$

Proof. The proof of this bound follows an argument in Bastian & Rivière (2003) and we present its main points. Let us denote

$$\boldsymbol{\chi} = \boldsymbol{u}_h^{n+1} + \kappa \nabla P_h^{n+1}.$$

Then, from (4.21a)–(4.21b), we have for any $K, K' \in \mathscr{E}_h$, and any $e \subset \partial K$,

$$\int_e \boldsymbol{\chi}|_K \cdot \boldsymbol{n}_e q_h = \frac{1}{2}\int_e \kappa(\nabla P_h^{n+1}|_K - \nabla P_h^{n+1}|_{K'}) \cdot \boldsymbol{n}_e q_h + \alpha_{p_\ell,e} h_e^{-1} \int_e \eta_{p_\ell,e}^n [P_h^{n+1}] q_h, \qquad e = \partial K \cap \partial K',$$

$$\int_e \boldsymbol{\chi}|_K \cdot \boldsymbol{n}_e q_h = 0, \qquad e \subset \partial\Omega.$$

Let us take $q_h = \boldsymbol{\chi} \cdot \boldsymbol{n}_e$ in the above; this is allowed because P_h^{n+1} is piecewise linear and κ is assumed to be piecewise constant (see H.5). For edges on the boundary, we have

$$\|\boldsymbol{\chi}|_K \cdot \boldsymbol{n}_e\|_{L^2(e)} = 0.$$

For interior edges, we apply Cauchy–Schwarz's inequality:

$$\|\boldsymbol{\chi}|_K \cdot \boldsymbol{n}_e\|_{L^2(e)} \leqslant C\|[\nabla P_h^{n+1}]\cdot \boldsymbol{n}_e\|_{L^2(e)} + Ch_e^{-1}\|[P_h^{n+1}]\|_{L^2(e)}.$$

We now bound $\|\boldsymbol{\chi}\|_{L^2(K)}$ by passing to the reference element, by using the fact that $\|\cdot\|_{L^2(\partial\hat{K})}$ is a norm for the Raviart–Thomas space restricted to $\hat{K}$ and by going back to the physical element:

$$\|\boldsymbol{\chi}\|_{L^2(K)} \leqslant Ch\|\hat{\boldsymbol{\chi}}\|_{L^2(\hat{K})} \leqslant Ch\|\hat{\boldsymbol{\chi}}\|_{L^2(\partial\hat{K})} \leqslant Ch^{1/2}\|\boldsymbol{\chi}\|_{L^2(\partial K)}.$$

We apply the bounds above:

$$\|\boldsymbol{\chi}\|_{L^2(K)} \leqslant Ch^{1/2}\sum_{e\in\partial E\setminus\partial\Omega}\|[\nabla P_h^{n+1}]\cdot \boldsymbol{n}_e\|_{L^2(e)} + C\sum_{e\in\partial E\setminus\partial\Omega} h^{-1/2}\|[P_h^{n+1}]\|_{L^2(e)}.$$

Taking the square and summing over all the elements,

$$\|\boldsymbol{\chi}\|^2 \leqslant Ch\sum_{K\in\mathscr{E}_h}\Big(\sum_{e\in\partial K\setminus\partial\Omega}\|[\nabla P_h^{n+1}]\cdot \boldsymbol{n}_e\|_{L^2(e)}\Big)^2 + C\sum_{K\in\mathscr{E}_h}\Big(\sum_{e\in\partial K\setminus\partial\Omega} h^{-1/2}\|[P_h^{n+1}]\|_{L^2(e)}\Big)^2.$$

The last term is bounded above by $|||P_h^{n+1} - p_\ell^{n+1}|||^2$ since $[p_\ell] = 0$. For the first term, we write for $e = \partial K \cap \partial K'$,

$$\|[\nabla P_h^{n+1}]\|_{L^2(e)} \leqslant \|[\nabla(P_h^{n+1} - p_\ell^{n+1})]\|_{L^2(e)}.$$

Clearly, we have

$$\|[\nabla(P_h^{n+1} - p_\ell^{n+1})]\|_{L^2(e)} \leqslant C(\|\nabla(P_h^{n+1} - p_\ell^{n+1})|_K\|_{L^2(e)} + \|\nabla(P_h^{n+1} - p_\ell^{n+1})|_{K'}\|_{L^2(e)}).$$

We add and subtract the L^2 projection of p_ℓ^{n+1} onto X_h:

$$\begin{aligned}\|\nabla(P_h^{n+1} - p_\ell^{n+1})|_K\|_{L^2(e)} &\leqslant \|\nabla(P_h^{n+1} - \pi_{h,\Gamma_D^{p_\ell}} p_\ell^{n+1})|_K\|_{L^2(e)} + \|\nabla(\pi_{h,\Gamma_D^{p_\ell}} p_\ell^{n+1} - p_\ell^{n+1})|_K\|_{L^2(e)}\\ &\leqslant Ch^{-1/2}\|\nabla(P_h^{n+1} - \pi_{h,\Gamma_D^{p_\ell}} p_\ell^{n+1})\|_{L^2(K)} + Ch^{1/2}\|p_\ell^{n+1}\|_{H^2(K)}.\end{aligned}$$

So

$$h\sum_{K\in\mathscr{E}_h}\Big(\sum_{e\in\partial K\setminus\partial\Omega}\|\nabla(P_h^{n+1} - p_\ell^{n+1})\|_{L^2(e)}\Big)^2 \leqslant C\sum_{K\in\mathscr{E}_h}\|\nabla(P_h^{n+1} - \pi_{h,\Gamma_D^{p_\ell}} p_\ell^{n+1})\|_{L^2(K)}^2 + Ch^2\|p_\ell^{n+1}\|_{H^2(\Omega)}^2,$$

or

$$\leqslant C|||P_h^{n+1} - p_\ell^{n+1}|||^2 + Ch^2\|p_\ell^{n+1}\|_{H^2(\Omega)}^2.$$

Combining all the bounds we have

$$\|\boldsymbol{\chi}\| \leqslant C\||P_h^{n+1} - p_\ell^{n+1}\|| + Ch. \qquad \square$$

5.2 *Hypotheses*

In the remainder of the paper, the following assumptions are made on the input data.

H.1 The nonlinear functions λ_i, for $i = v, \ell, a$, are C^2 functions with respect to time. Moreover, we have the following bounds:

$$\begin{array}{lllllll} 0 & < & \underline{C}_{(\rho\lambda)_t} & \leqslant & \Big(\rho\lambda\Big)_t & \leqslant & \overline{C}_{(\rho\lambda)_t}, \\ 0 & < & \underline{C}_{\lambda_i} & \leqslant & \lambda_i & \leqslant & \overline{C}_{\lambda_i}, \\ 0 & < & \underline{C}_{\lambda_t} & \leqslant & \lambda_t & \leqslant & \overline{C}_{\lambda_t}, \\ 0 & < & \kappa_* & \leqslant & \kappa & \leqslant & \kappa^*, \\ 0 & < & \underline{C}_{p_{c,a}} & \leqslant & \Big(\partial_{s_a} p_{c,a}\Big)^+ & \leqslant & \overline{C}_{p_{c,a}}, \\ 0 & \leqslant & \underline{C}_{p_{c,v}} & \leqslant & \partial_{s_v} p_{c,v} & \leqslant & \overline{C}_{p_{c,v}}. \end{array} \tag{5.11}$$

REMARK 5.5 We note that the above bounds also hold when these functions are evaluated with discrete solutions by using cutoff in the definition of the above functions.

H.2 The following functions are Lipschitz continuous, so that we have

$$\begin{aligned} &|\lambda_i(s_{a_1}, s_{v_1}) - \lambda_i(s_{a_2}, s_{v_2})| \leqslant L\Big(|s_{a_1} - s_{a_2}| + |s_{v_1} - s_{v_2}|\Big), \\ &|\partial_{s_v} p_{c,v}(s_{v_1}) - \partial_{s_v} p_{c,v}(s_{v_2})| \leqslant L|s_{v_1} - s_{v_2}|, \\ &|\partial_{s_v} p_{c,a}(s_{a_1}) - \partial_{s_v} p_{c,a}(s_{a_2})| \leqslant L|s_{a_1} - s_{a_2}|. \end{aligned} \tag{5.12}$$

H.3 The functions $\nabla p_{c,a}$ and $\nabla p_{c,v}$ are bounded, so that we have

$$\begin{array}{lll} 0 \leqslant \underline{C}_{\nabla p_{c_a}} & \leqslant \|\nabla p_{c_a}\|_{L^\infty(\Omega)} & \leqslant \overline{C}_{\nabla p_{c_a}}, \\ 0 \leqslant \underline{C}_{\nabla p_{c_v}} & \leqslant \|\nabla p_{c_v}\|_{L^\infty(\Omega)} & \leqslant \overline{C}_{\nabla p_{c_v}}, \end{array} \tag{5.13}$$

and they satisfy the growth conditions

$$\begin{array}{lll} \|\nabla p_{c_a}(s_{a_1}) - \nabla p_{c_a}(s_{a_2})\| & \leqslant & L\|s_{a_1} - s_{a_2}\|, \\ \|\nabla p_{c_v}(s_{v_1}) - \nabla p_{c_v}(s_{v_2})\| & \leqslant & L\|s_{v_1} - s_{v_2}\|. \end{array} \tag{5.14}$$

We remark that, even though this hypothesis might be somewhat restrictive, it has been used before in e.g., Chen & Ewing (2001), Radu *et al.* (2018). For instance, in Chen & Ewing (2001), assumptions (A5) and (A7) state that the functions γ_1 and γ_2, which contain the gradient of the capillary pressure, are bounded and Lipschitz continuous with respect to the primary unknown θ.

H.4 The source terms q_i are smooth enough: $q_i \in L^\infty(0, T; L^\infty(\Omega))$, for $i = \ell, v, a$.

H.5 The absolute permeability κ is piecewise constant.

6. Existence and uniqueness

In the following we denote by p_ℓ, s_a and s_v the exact solutions to (2.6), (2.8) and (2.9). We assume that the exact solutions are smooth enough, more precisely $p_\ell, s_v, s_a \in C^2(0,T;L^2(\Omega)) \cap C^0(0,T;H^2(\Omega)) \cap L^\infty(0,T;W^{1,\infty}(\Omega))$.

For readability, we denote by $\tilde{\lambda}_t, \tilde{p}_{c,a}, \tilde{p}_{c,v}, \tilde{\lambda}_i$ for $i = v, \ell, a$ the functions $\lambda_t, p_{c,a}, p_{c,v}, \lambda_i$ evaluated at the exact solutions (pressures and saturations) at time t. If the time is t_n, then the functions are denoted by $\tilde{\lambda}_t^n, \tilde{p}_{c,a}^n, \tilde{p}_{c,v}^n, \tilde{\lambda}_i^n$. For instance, we will write

$$\tilde{\lambda}_a^n = \lambda_a(s_a^n, s_v^n), \quad \lambda_a^n = \lambda_a(S_{a_h}^n, S_{v_h}^n).$$

Existence and uniqueness of $P_h^{n+1}, S_{a_h}^{n+1}, S_{v_h}^{n+1}$ follow from the linearity of (4.6), (4.11), (4.16) with respect to their unknowns and from the coercivity and continuity of the forms b_p, b_a and b_v.

6.1 *Liquid pressure*

Lemma 6.1 (Consistency of b_p). We have for any $n \geqslant 0$ and any $w_h \in X_{h,\Gamma_{\mathrm{D}}^{p_\ell}}$,

$$\tilde{b}_p^{n+1}(p_\ell^{n+1}, w_h) = \tilde{f}_p^{n+1}(w_h), \tag{6.1}$$

where

$$\tilde{b}_p^{n+1}(p_\ell, w_h) = b_p(p_\ell, w_h; p_\ell^{n+1}, s_a^{n+1}, s_v^{n+1}) \quad \text{and} \quad \tilde{f}_p^{n+1}(w_h) = f_p(w_h; p_\ell^{n+1}, s_a^{n+1}, s_v^{n+1}). \tag{6.2}$$

Proof. First, note that

$$\tilde{b}_p^{n+1}(p_\ell, w_h) = \sum_{K\in\mathscr{E}_h} \int_K \tilde{\lambda}_t^{n+1} \kappa \nabla p_\ell \cdot \nabla w_h - \sum_{e\in\Gamma_h} \int_e \tilde{\lambda}_t^{n+1} \kappa \nabla p_\ell \cdot \boldsymbol{n}_e \left[w_h\right]. \tag{6.3}$$

In the rest of the proof, we drop the superscript $(n+1)$ for readability, but it is understood that all functions are evaluated at time t_{n+1}. Applying integration by parts on the first term, we obtain

$$\tilde{b}_p(p_\ell, w_h) = -\sum_{K\in\mathscr{E}_h} \int_K \nabla \cdot \left(\tilde{\lambda}_t \kappa \nabla p_\ell\right) w_h + \sum_{K\in\mathscr{E}_h} \int_{\partial K} \tilde{\lambda}_t \kappa \nabla p_\ell \cdot \boldsymbol{n}_K w_h - \sum_{e\in\Gamma_h} \int_e \tilde{\lambda}_t \kappa \nabla p_\ell \cdot \boldsymbol{n}_e \left[w_h\right]. \tag{6.4}$$

Using the fact that $\left[\tilde{\lambda}_t \kappa \nabla p_\ell \cdot \boldsymbol{n}_e\right] = 0$ on interior faces, we obtain

$$\tilde{b}_p(p_\ell, w_h) = -\sum_{K\in\mathscr{E}_h} \int_K \nabla \cdot \left(\tilde{\lambda}_t \kappa \nabla p_\ell\right) w_h + \sum_{e\in\Gamma_{\mathrm{N}}^{p_\ell}} \int_e \tilde{\lambda}_t \kappa \nabla p_\ell \cdot \boldsymbol{n}_e w_h. \tag{6.5}$$

On the other hand, after integration by parts on the volume term of (4.8), we have

$$\begin{aligned}\tilde{f}_p(w_h) = (q_t, w_h) &+ \sum_{K\in\mathscr{E}_h}\int_K \nabla\cdot\Big(\tilde{\lambda}_v\kappa\nabla\tilde{p}_{c,v} - \tilde{\lambda}_a\kappa\nabla\tilde{p}_{c,a} - \kappa\big(\rho\tilde{\lambda}\big)_t\boldsymbol{g}\Big)w_h\\ &- \sum_{K\in\mathscr{E}_h}\int_{\partial K}\Big(\tilde{\lambda}_v\kappa\nabla\tilde{p}_{c,v} - \tilde{\lambda}_a\kappa\nabla\tilde{p}_{c,a} - \kappa\big(\rho\tilde{\lambda}\big)_t\boldsymbol{g}\Big)\cdot\boldsymbol{n}_K w_h\\ &+ \sum_{e\in\Gamma_{\mathrm{N}}^{p_\ell}}\int_e j_p^{\mathrm{N}} w_h + \sum_{e\in\Gamma_h}\int_e\Big\{\tilde{\lambda}_v\kappa\nabla\tilde{p}_{c,v}\cdot\boldsymbol{n}_e\Big\}\Big[w_h\Big] - \sum_{e\in\Gamma_h}\int_e\Big\{\tilde{\lambda}_a\kappa\nabla\tilde{p}_{c,a}\cdot\boldsymbol{n}_e\Big\}\Big[w_h\Big]\\ &- \sum_{e\in\Gamma_h}\int_e\Big\{\kappa\big(\rho\tilde{\lambda}\big)_t\boldsymbol{g}\cdot\boldsymbol{n}_e\Big\}\Big[w_h\Big].\end{aligned} \tag{6.6}$$

Using that $\Big[\big(\tilde{\lambda}_v\kappa\nabla\tilde{p}_{c,v} - \tilde{\lambda}_a\kappa\nabla\tilde{p}_{c,a} - \kappa\big(\rho\tilde{\lambda}\big)_t\boldsymbol{g}\big)\cdot\boldsymbol{n}\Big] = 0$ on interior faces, we obtain

$$\begin{aligned}\tilde{f}_p(w_h) = (q_t, w_h) &+ \sum_{K\in\mathscr{E}_h}\int_K \nabla\cdot\Big(\tilde{\lambda}_v\kappa\nabla\tilde{p}_{c,v} - \tilde{\lambda}_a\kappa\nabla\tilde{p}_{c,a} - \kappa\big(\rho\tilde{\lambda}\big)_t\boldsymbol{g}\Big)w_h\\ - \sum_{e\in\Gamma_{\mathrm{N}}^{p_\ell}}\int_e\tilde{\lambda}_v\kappa\nabla\tilde{p}_{c,v}\cdot\boldsymbol{n}_e w_h &+ \sum_{e\in\Gamma_{\mathrm{N}}^{p_\ell}}\int_e\tilde{\lambda}_a\kappa\nabla\tilde{p}_{c,a}\cdot\boldsymbol{n}_e w_h + \sum_{e\in\Gamma_{\mathrm{N}}^{p_\ell}}\int_e\kappa\big(\rho\tilde{\lambda}\big)_t\boldsymbol{g}\cdot\boldsymbol{n}_e w_h + \sum_{e\in\Gamma_{\mathrm{N}}^{p_\ell}}\int_e j_p^{\mathrm{N}} w_h.\end{aligned} \tag{6.7}$$

Recalling that p_ℓ solves (2.6) and satisfies the boundary condition (2.11a), the result (6.1) follows. □

LEMMA 6.2 For all $(v_h, w_h)\in X_h\times X_h$, the following relation is satisfied for all $n\geqslant 0$:

$$\Big|\sum_{e\in\Gamma_h}\int_e\Big\{\lambda_t^n\kappa\nabla v_h\cdot\boldsymbol{n}_e\Big\}\Big[w_h\Big]\Big| \leqslant \overline{C}_{\lambda_t}\kappa^*\Big(\sum_{K\in\mathscr{E}_h}\sum_{e\in\partial K} h_e\|\nabla v_h|_K\cdot\boldsymbol{n}_e\|^2_{L^2(e)}\Big)^{1/2}|w_h|_{\mathrm{J}}. \tag{6.8}$$

Proof. Let us consider a face $e\in\Gamma_h$ that is shared between two elements K_1 and K_2, i.e., $e=\partial K_1\cap\partial K_2$. With H.1 and Cauchy–Schwarz's inequality, we have

$$\int_e\Big\{\lambda_t^n\kappa\nabla v_h\cdot\boldsymbol{n}_e\Big\}\Big[w_h\Big] \leqslant \overline{C}_{\lambda_t}\kappa^* h_e^{1/2}\Big(\Big\|\nabla v_h|_{K_1}\cdot\boldsymbol{n}_e\Big\|_{L^2(e)} + \Big\|\nabla v_h|_{K_2}\cdot\boldsymbol{n}_e\Big\|_{L^2(e)}\Big)h_e^{-1/2}\Big\|\Big[w_h\Big]\Big\|_{L^2(e)}. \tag{6.9}$$

Summing over all faces, applying Cauchy–Schwarz's inequality and writing the sum in terms of the face contributions for each element, we obtain the result. □

Next we show that b_p is coercive on $X_{h,\Gamma_{\mathrm{D}}^{p_\ell}}$.

LEMMA 6.3 (Coercivity of b_p). Assume that $\alpha_{p_\ell,*}$ satisfies

$$\alpha_{p_\ell,*} > 0.25\Big(1-\theta_{p_\ell}\Big)^2\Big(\overline{C}_{\lambda_t}\kappa^*\Big)^3\Big(\underline{C}_{\lambda_t}\kappa_*\Big)^{-3}C_{\mathrm{tr}}^2, \tag{6.10}$$

where C_{tr} results from the trace inequality (5.3). Then the bilinear form b_p^n defined by (4.7) is coercive on X_h with respect to the norm $|||\cdot|||$ defined by (5.1), i.e., for all $w_h \in X_{h,\Gamma_{\mathrm{D}}^{p_\ell}}$ and for all $n \geqslant 0$, the following relation is satisfied:

$$b_p^n(w_h, w_h) \geqslant C_{\alpha,p_\ell}|||w_h|||^2, \tag{6.11}$$

with

$$C_{\alpha,p_\ell} = \frac{\alpha_{p_\ell,*}\underline{C}_{\lambda_t}\kappa_*\Big(\overline{C}_{\lambda_t}\kappa^*\Big)^{-1} - 0.25\Big(1-\theta_{p_\ell}\Big)^2\Big(\overline{C}_{\lambda_t}\kappa^*\Big)^2\Big(\underline{C}_{\lambda_t}\kappa_*\Big)^{-2}C_{\mathrm{tr}}^2}{1+\alpha_{p_\ell,*}\underline{C}_{\lambda_t}\kappa_*\Big(\overline{C}_{\lambda_t}\kappa^*\Big)^{-1}}. \tag{6.12}$$

Proof. Using (4.7) we have

$$\begin{aligned} b_p^n(w_h, w_h) &= \sum_{K\in\mathscr{E}_h}\int_K \lambda_t^n\kappa\Big|\nabla w_h\Big|^2 + \sum_{e\in\Gamma_h}\alpha_{p_\ell,e}h_e^{-1}\int_e \eta_{p_\ell,e}^n[w_h]^2 + (\theta_{p_\ell}-1)\sum_{e\in\Gamma_h}\int_e\Big\{\lambda_t^n\kappa\nabla w_h\cdot\boldsymbol{n}_e\Big\}\Big[w_h\Big] \\ &\geqslant \underline{C}_{\lambda_t}\kappa_*\|\nabla w_h\|^2 + \alpha_{p_\ell,*}\frac{\underline{C}_{\lambda_t}^2\kappa_*^2}{\overline{C}_{\lambda_t}\kappa^*}|w_h|_{\mathrm{J}}^2 + \Big(\theta_{p_\ell}-1\Big)\sum_{e\in\Gamma_h}\int_e\Big\{\lambda_t^n\kappa\nabla w_h\cdot\boldsymbol{n}_e\Big\}\Big[w_h\Big]. \end{aligned} \tag{6.13}$$

Using (6.8), the trace inequality (5.3) and the fact that for all $K\in\mathscr{E}_h$ and all $e\in\partial K$, $h_e \leqslant h_K$, we have

$$\begin{aligned} \left|\sum_{e\in\Gamma_h}\int_e\Big\{\lambda_t^n\kappa\nabla w_h\cdot\boldsymbol{n}_e\Big\}\Big[w_h\Big]\right| &\leqslant \overline{C}_{\lambda_t}\kappa^*\left(\sum_{K\in\mathscr{E}_h}\sum_{e\in\partial K}h_e\|\nabla w_h|_K\cdot\boldsymbol{n}_e\|_{L^2(e)}^2\right)^{1/2}|w_h|_{\mathrm{J}} \\ &\leqslant \overline{C}_{\lambda_t}\kappa^*C_{\mathrm{tr}}\|\nabla w_h\||w_h|_{\mathrm{J}}. \end{aligned} \tag{6.14}$$

Thus, since $\theta_{p_\ell}-1\leqslant 0$, we have

$$\Big(\theta_{p_\ell}-1\Big)\sum_{e\in\Gamma_h}\int_e\Big\{\lambda_t^n\kappa\nabla w_h\cdot\boldsymbol{n}_e\Big\}\Big[w_h\Big] \geqslant \Big(\theta_{p_\ell}-1\Big)\overline{C}_{\lambda_t}\kappa^*C_{\mathrm{tr}}\|\nabla w_h\||w_h|_{\mathrm{J}}. \tag{6.15}$$

Using this in (6.13) and noting that $\theta_{p_\ell}-1$ is equal to either -2, -1 or 0, we have

$$b_p^n(w_h, w_h) \geqslant \underline{C}_{\lambda_t}\kappa_*\|\nabla w_h\|^2 + \alpha_{p_\ell,*}\frac{\underline{C}_{\lambda_t}^2\kappa_*^2}{\overline{C}_{\lambda_t}\kappa^*}|w_h|_{\mathrm{J}}^2 - 2\frac{1-\theta_{p_\ell}}{2}\overline{C}_{\lambda_t}\kappa^*C_{\mathrm{tr}}\|\nabla w_h\||w_h|_{\mathrm{J}}. \tag{6.16}$$

Next we use the following inequality: let β be a non-negative real number and assume that $c > \beta^2$; then, for all $x, y \in \mathbb{R}$,

$$x^2 - 2\beta xy + cy^2 \geqslant \frac{c-\beta^2}{1+c}\Big(x^2 + y^2\Big). \tag{6.17}$$

Using this in (6.16) with $c = \alpha_{p_\ell,*}\underline{C}_{\lambda_t}\kappa_*\big(\overline{C}_{\lambda_t}\kappa^*\big)^{-1}$, $\beta = 0.5\big(1-\theta_{p_\ell}\big)C_{\mathrm{tr}}\overline{C}_{\lambda_t}\kappa^*\big(\underline{C}_{\lambda_t}\kappa_*\big)^{-1}$, $x = \big(\underline{C}_{\lambda_t}\kappa_*\big)^{1/2}\|\nabla w_h\|$ and $y = \big(\underline{C}_{\lambda_t}\kappa_*\big)^{1/2}|w_h|_{\mathrm{J}}$ concludes the proof. □

Now we prove that b_p is bounded.

LEMMA 6.4 (Boundedness of b_p). There exists a constant $C_{\mathrm{B},p_\ell} > 0$ independent of h such that, for all $v_h \in X_{h,\Gamma_{\mathrm{D}}^{p_\ell}}$ and $w_h \in X_{h,\Gamma_{\mathrm{D}}^{p_\ell}}$, the following relation is satisfied,

$$b_p^n(v_h, w_h) \leqslant C_{\mathrm{B},p_\ell}|||v_h|||\ |||w_h|||. \tag{6.18}$$

In addition, there exists a constant $C_{\mathrm{B}_*,p_\ell} > 0$ independent of h and τ such that for any $v \in H^2(\Omega) + X_{h,\Gamma_{\mathrm{D}}^{p_\ell}}$ and any $w_h \in X_{h,\Gamma_{\mathrm{D}}^{p_\ell}}$, the following bound holds:

$$|b_p^n(v, w_h)| \leqslant C_{\mathrm{B}_*,p_\ell}|||v|||_*|||w_h|||. \tag{6.19}$$

Proof. Let $v_h \in X_{h,\Gamma_{\mathrm{D}}^{p_\ell}}$ and $w_h \in X_{h,\Gamma_{\mathrm{D}}^{p_\ell}}$. We have

$$\begin{aligned}|b_p^n(v_h, w_h)| \leqslant{}& \left|\sum_{K\in\mathscr{E}_h}\int_K \lambda_t^n\kappa\nabla v_h\cdot\nabla w_h\right| + \left|\sum_{e\in\Gamma_h}\alpha_{p_\ell,e}h_e^{-1}\int_e \eta_{p_\ell,e}^n\Big[v_h\Big]\Big[w_h\Big]\right| \\ &+ \left|\sum_{e\in\Gamma_h}\int_e\Big\{\lambda_t^n\kappa\nabla v_h\cdot\boldsymbol{n}_e\Big\}\Big[w_h\Big]\right| + \left|\sum_{e\in\Gamma_h}\theta_{p_\ell}\int_e\Big\{\lambda_t^n\kappa\nabla w_h\cdot\boldsymbol{n}_e\Big\}\Big[v_h\Big]\right| \\ ={}& T_1 + T_2 + T_3 + T_4.\end{aligned} \tag{6.20}$$

Using Cauchy–Schwarz's inequality, we see that

$$T_1 \leqslant \overline{C}_{\lambda_t}\kappa^*\|\nabla_h v_h\|\|\nabla_h w_h\| \leqslant \overline{C}_{\lambda_t}\kappa^*|||v_h|||\ |||w_h|||. \tag{6.21}$$

Similarly,

$$T_2 \leqslant \alpha_{p_\ell}^*\frac{\Big(\overline{C}_{\lambda_t}\kappa^*\Big)^2}{\underline{C}_{\lambda_t}\kappa_*}|v_h|_{\mathrm{J}}|w_h|_{\mathrm{J}} \leqslant \alpha_{p_\ell}^*\frac{\Big(\overline{C}_{\lambda_t}\kappa^*\Big)^2}{\underline{C}_{\lambda_t}\kappa_*}|||v_h|||\ |||w_h|||. \tag{6.22}$$

Using (6.8), recalling that $h_e \leqslant h_K$ and using the trace inequality (5.3), we bound T_3 as

$$T_3 \leqslant \overline{C}_{\lambda_t}\kappa^* |||v_h||| \; |||w_h|||. \tag{6.23}$$

Finally, T_4 can be bounded in a similar way as T_3 to obtain

$$T_4 \leqslant |\theta_{p_\ell}|\overline{C}_{\lambda_t}\kappa^* |||v_h||| \; |||w_h|||. \tag{6.24}$$

Taking $C_{\mathrm{B},p_\ell} = 4\overline{C}_{\lambda_t}\sqrt{\kappa^*}\max\left(\alpha^*_{p_\ell}\frac{\overline{C}_{\lambda_t}\kappa^*}{\underline{C}_{\lambda_t}\kappa_*}, |\theta_{p_\ell}|\right)$ gives the result. The proof of (6.19) is similar; one needs to change the bound for the term T_3. □

Corollary 6.5 There exists a unique solution to problem (4.6).

Proof. The coercivity Lemma 6.3 and boundedness Lemma 6.4 of b_p, together with the fact that ψ_ℓ is strictly positive, imply, using the Lax–Milgram theorem, that problem (4.6) is well posed. □

6.2 *Aqueous saturation*

Lemma 6.6 (Consistency of b_a).

$$(\phi(\partial_t s_a)^{n+1}, w_h) + \tilde{b}_a^{n+1}(s_a^{n+1}, w_h) = \tilde{f}_a^{n+1}(w_h) \quad \forall\, w_h \in X_{h,\Gamma_{\mathrm{D}}^{s_a}}, \quad \forall\, n \geqslant 0, \tag{6.25}$$

where $\tilde{b}_a^{n+1}(s_a, w_h) = b_a(s_a, w_h; p_\ell^{n+1}, s_a^{n+1}, s_v^{n+1})$ and $\tilde{f}_a^{n+1}(w_h) = f_a(w_h; p_\ell^{n+1}, s_a^{n+1}, s_v^{n+1})$.

Proof. The proof of this lemma is skipped because it is analogous to the proof of Lemma 6.1. □

Lemma 6.7 For all $(v_h, w_h) \in X_h \times X_h$, the following relation is satisfied:

$$\begin{aligned}&\left|\sum_{e\in\Gamma_h}\int_e \left\{\kappa\lambda_a^n\left(\partial_{s_a}p_{c,a}\right)^{+,n}\nabla v_h\cdot \boldsymbol{n}_e\right\}\left[w_h\right]\right|\\ &\quad\leqslant \overline{C}_{\lambda_a}\kappa^*\overline{C}_{p_{c,a}}\left(\sum_{K\in\mathscr{E}_h}\sum_{e\in\partial K} h_e\|\nabla v_h|_K\cdot\boldsymbol{n}_e\|^2_{L^2(e)}\right)^{1/2} |w_h|_{\mathrm{J}}.\end{aligned} \tag{6.26}$$

Proof. The proof of this lemma is analogous to the proof of (6.8). □

Now we can show that b_a is coercive on $X_{h,\Gamma_{\mathrm{D}}^{s_a}}$.

Lemma 6.8 (Coercivity of b_a). Assume that $\alpha_{s_a,*}$ satisfies

$$\alpha_{s_a,*} > 0.25\left(1-\theta_{s_a}\right)^2\left(\overline{C}_{\lambda_a}\overline{C}_{p_{c,a}}\kappa^*\right)^3\left(\underline{C}_{\lambda_a}\underline{C}_{p_{c,a}}\kappa_*\right)^{-3}C_{\mathrm{tr}}^2, \tag{6.27}$$

where C_{tr} results from the trace inequality (5.3). Then the bilinear form b_a^n defined by (4.12) is coercive on $X_{h,\Gamma_{\mathrm{D}}^{s_a}}$ with respect to the norm $|||\cdot|||$ defined by (5.1), i.e., for all $w_h \in X_{h,\Gamma_{\mathrm{D}}^{s_a}}$, the following relation is satisfied:

$$b_a^n(w_h, w_h) \geqslant C_{\alpha,s_a} |||w_h|||^2, \tag{6.28}$$

with

$$C_{\alpha,s_a} = \frac{\alpha_{s_a,*}\underline{C}_{p_{c,a}}\underline{C}_{\lambda_a}\kappa_*\left(\overline{C}_{p_{c,a}}\overline{C}_{\lambda_a}\kappa^*\right)^{-1} - 0.25\left(1-\theta_{s_a}\right)^2\left(\overline{C}_{p_{c,a}}\overline{C}_{\lambda_a}\kappa^*\right)^2\left(\underline{C}_{p_{c,a}}\underline{C}_{\lambda_a}\kappa_*\right)^{-2}C_{\mathrm{tr}}^2}{1+\alpha_{s_a,*}\underline{C}_{p_{c,a}}\underline{C}_{\lambda_a}\kappa_*\left(\overline{C}_{p_{c,a}}\overline{C}_{\lambda_a}\kappa^*\right)^{-1}}. \tag{6.29}$$

Proof. Using (4.12), (6.26), the trace inequality (5.3), the fact that for all $K \in \mathscr{E}_h$ and all $e \in \partial K$, $h_e \leqslant h_K$, and that $\theta_{s_a} - 1 \leqslant 0$ we have

$$\begin{aligned} b_a^n(w_h, w_h) &= \sum_{K\in\mathscr{E}_h}\int_K \kappa\lambda_a^n\left(\partial_{s_a}p_{c,a}\right)^{+,n}|\nabla w_h|^2 + \sum_{e\in\Gamma_h}\alpha_{s_a,e}h_e^{-1}\int_e \eta_{s_a,e}^n\left(\left[w_h\right]\right)^2 \\ &\quad + (\theta_{s_a}-1)\sum_{e\in\Gamma_h}\int_e\left\{\kappa\lambda_a^n\left(\partial_{s_a}p_{c,a}\right)^{+,n}\nabla w_h\cdot\boldsymbol{n}_e\right\}\left[w_h\right] \\ &\geqslant \underline{C}_{p_{c,a}}\kappa_*\underline{C}_{\lambda_a}\|\nabla w_h\|^2 + \alpha_{s_a,*}\frac{\underline{C}_{p_{c,a}}^2\underline{C}_{\lambda_a}^2\kappa_*^2}{\overline{C}_{p_{c,a}}\overline{C}_{\lambda_a}\kappa^*}|w_h|_{\mathrm{J}}^2 + (\theta_{s_a}-1)\overline{C}_{\lambda_a}\kappa^*\overline{C}_{p_{c,a}}C_{\mathrm{tr}}\|\nabla w_h\||w_h|_{\mathrm{J}}. \end{aligned} \tag{6.30}$$

Using (6.17) with $c = \alpha_{s_a,*}\underline{C}_{p_{c,a}}\underline{C}_{\lambda_a}\kappa_*\left(\overline{C}_{p_{c,a}}\overline{C}_{\lambda_a}\kappa^*\right)^{-1}$, $\beta = 0.5(1-\theta_{s_a})\overline{C}_{\lambda_a}\kappa^*\overline{C}_{p_{c,a}}C_{\mathrm{tr}}\left(\underline{C}_{p_{c,a}}\kappa_*\underline{C}_{\lambda_a}\right)^{-1}$, $x = \left(\underline{C}_{p_{c,a}}\kappa_*\underline{C}_{\lambda_a}\right)^{1/2}\|\nabla_h w_h\|$ and $y = \left(\underline{C}_{p_{c,a}}\kappa_*\underline{C}_{\lambda_a}\right)^{1/2}|w_h|_{\mathrm{J}}$ concludes the proof. □

Now we prove that b_a is bounded.

LEMMA 6.9 (Boundedness of b_a). There exists a constant $C_{\mathrm{B},s_a} > 0$ independent of h such that, for all $n \geqslant 0$, $v_h \in X_{h,\Gamma_{\mathrm{D}}^{s_a}}$ and $w_h \in X_{h,\Gamma_{\mathrm{D}}^{s_a}}$, the following relation is satisfied:

$$|b_a^n(v_h, w_h)| \leqslant C_{\mathrm{B},s_a}|||v_h|||\ |||w_h|||. \tag{6.31}$$

In addition, there exists a constant $C_{\mathrm{B}_*,s_a} > 0$ independent of h and τ such that for any $v \in H^2(\Omega) + X_{h,\Gamma_{\mathrm{D}}^{s_a}}$ and any $w_h \in X_{h,\Gamma_{\mathrm{D}}^{s_a}}$, the following bound holds:

$$|b_a^n(v, w_h)| \leqslant C_{\mathrm{B}_*,s_a}|||v|||_*|||w_h|||. \tag{6.32}$$

Proof. The proof of this lemma is analogous to that of Lemma 6.4 and therefore is omitted. □

COROLLARY 6.10 There exists a unique solution to problem (4.11).

Proof. The coercivity Lemma 6.7 and boundedness Lemma 6.8 of a_{s_a}, together with the fact that ϕ and ρ_a are strictly positive, imply, using the Lax–Milgram theorem, that problem (4.11) is well posed. □

6.3 *Vapor saturation*

LEMMA 6.11 (Consistency of b_v).

$$(\phi(\partial_t s_v)^{n+1}, w_h) + \tilde{b}_v^{n+1}(s_v^{n+1}, w_h) = \tilde{f}_v^{n+1}(w_h), \quad \forall\, w_h \in X_h, \quad \forall\, n \geqslant 0, \tag{6.33}$$

where $\tilde{b}_v^{n+1}(s_v, w_h) = b_v(s_v, w_h; p_\ell^{n+1}, s_a^{n+1}, s_v^{n+1})$ and $\tilde{f}_v^{n+1}(w_h) = f_v(w_h; p_\ell^{n+1}, s_a^{n+1}, s_v^{n+1})$.

Proof. The proof is similar to the other consistency lemma and therefore not shown here. □

LEMMA 6.12 For all $(v_h, w_h) \in X_h \times X_h$ and any $n \geqslant 0$, the following relation is satisfied:

$$\begin{aligned} &\left| \sum_{e\in\Gamma_h} \int_e \left\{ \kappa \lambda_v^n \left(\partial_{s_v} p_{c,v}\right)^n \nabla v_h \cdot \boldsymbol{n}_e \right\} \left[w_h \right] \right| \\ &\leqslant \overline{C}_{\lambda_v} \kappa^* \overline{C}_{p_{c,v}} \left(\sum_{K\in\mathscr{E}_h} \sum_{e\in\partial K} h_e \| \nabla v_h|_K \cdot \boldsymbol{n}_e \|^2_{L^2(e)} \right)^{1/2} |w_h|_{\mathrm{J}}. \end{aligned} \tag{6.34}$$

Proof. The proof of this lemma is completely analogous to the proof of (6.8). □

Now we can show that b_v is coercive on X_h.

LEMMA 6.13 (Coercivity of b_v). Assume that $\alpha_{s_v,*}$ satisfies

$$\alpha_{s_v,*} > 0.25\left(1-\theta_{s_v}\right)^2 \left(\overline{C}_{\lambda_v} \overline{C}_{p_{c,v}} \kappa^*\right)^3 \left(\underline{C}_{\lambda_v} \underline{C}_{p_{c,v}} \kappa_*\right)^{-3} C_{\mathrm{tr}}^2, \tag{6.35}$$

where C_{tr} results from the trace inequality (5.3). Then the bilinear form b_v^n defined by (4.17) is coercive on $X_{h,\Gamma_\mathrm{D}^{s_v}}$ with respect to the norm $|||\cdot|||$ defined by (5.1), i.e., for all $w_h \in X_{h,\Gamma_\mathrm{D}^{s_v}}$, the following relation is satisfied:

$$b_v^n(w_h, w_h) \geqslant C_{\alpha,s_v} |||w_h|||, \tag{6.36}$$

with

$$C_{\alpha,s_v} = \frac{\alpha_{s_v,*} \underline{C}_{p_{c,v}} \underline{C}_{\lambda_v} \kappa_* \left(\overline{C}_{p_{c,v}} \overline{C}_{\lambda_v} \kappa^*\right)^{-1} - 0.25\left(1-\theta_{s_v}\right)^2 \left(\overline{C}_{p_{c,v}} \overline{C}_{\lambda_v} \kappa^*\right)^2 \left(\underline{C}_{p_{c,v}} \underline{C}_{\lambda_v} \kappa_*\right)^{-2} C_{\mathrm{tr}}^2}{1 + \alpha_{s_v,*} \underline{C}_{p_{c,v}} \underline{C}_{\lambda_v} \kappa_* \left(\overline{C}_{p_{c,v}} \overline{C}_{\lambda_v} \kappa^*\right)^{-1}}. \tag{6.37}$$

Proof. The proof is analogous to that of (6.28) and therefore not shown here. □

Now we prove that b_v is bounded.

LEMMA 6.14 (Boundedness of b_v). There exists a constant $C_{\mathrm{B},s_v} > 0$ independent of h such that, for all $n \geqslant 0$, $v_h \in X_{h,\Gamma_{\mathrm{D}}^{s_v}}$ and $w_h \in X_{h,\Gamma_{\mathrm{D}}^{s_v}}$, the following relation is satisfied:

$$|b_v^n(v_h, w_h)| \leqslant C_{\mathrm{B},s_v} |||v_h||| \; |||w_h|||. \tag{6.38}$$

In addition, there exists a constant $C_{\mathrm{B}_*,s_v} > 0$ independent of h and τ such that for any $v \in H^2(\Omega) + X_{h,\Gamma_{\mathrm{D}}^{s_v}}$ and any $w_h \in X_{h,\Gamma_{\mathrm{D}}^{s_v}}$, the following bound holds:

$$|b_v^n(v, w_h)| \leqslant C_{\mathrm{B}_*,s_v} |||v|||_* |||w_h|||. \tag{6.39}$$

Proof. The proof of this lemma follows that of Lemma 6.4 and therefore is not shown here. □

COROLLARY 6.15 There exists a unique solution to the discrete problem (4.16).

Proof. The coercivity Lemma 6.11 and boundedness Lemma 6.12 of a_{s_v}, together with the fact that ϕ and ρ_v are strictly positive, imply, using the Lax–Milgram theorem, that problem (4.16) is well posed. □

7. *A priori* error estimates

In this section we derive *a priori* error estimates. To do so, we introduce the following quantities:

$$e_{p_h}^n = P_h^n - \pi_{h,\Gamma_{\mathrm{D}}^{p_\ell}} p_\ell^n, \quad e_{p_\pi}^n = p_\ell^n - \pi_{h,\Gamma_{\mathrm{D}}^{p_\ell}} p_\ell^n, \tag{7.1a}$$

$$e_{a_h}^n = S_{a_h}^n - \pi_{h,\Gamma_{\mathrm{D}}^{s_a}} s_a^n, \quad e_{a_\pi}^n = s_a^n - \pi_{h,\Gamma_{\mathrm{D}}^{s_a}} s_a^n, \tag{7.1b}$$

$$e_{v_h}^n = S_{v_h}^n - \pi_{h,\Gamma_{\mathrm{D}}^{s_v}} s_v^n, \quad e_{v_\pi}^n = s_v^n - \pi_{h,\Gamma_{\mathrm{D}}^{s_v}} s_v^n. \tag{7.1c}$$

We can then decompose the errors as

$$p_\ell^n - P_h^n = e_{p_\pi}^n - e_{p_h}^n, \tag{7.2a}$$

$$s_a^n - S_{a_h}^n = e_{p_\pi}^n - e_{p_h}^n, \tag{7.2b}$$

$$s_a^n - S_{a_h}^n = e_{p_\pi}^n - e_{p_h}^n. \tag{7.2c}$$

We note that, thanks to the definition of the L^2-orthogonal projection, the errors above satisfy

$$(e_{p_\pi}^n, w_h) = (e_{a_\pi}^n, w_h) = (e_{v_\pi}^n, w_h) = 0 \quad \forall\, w_h \in X_h. \tag{7.3}$$

We will make use of the following two auxiliary lemmas.

LEMMA 7.1 For any $0 \leqslant n \leqslant N-1$, and any $w_h \in X_h$, we have the following bounds:

$$\left|\tilde{b}_p^{n+1}(p_\ell^{n+1}, w_h) - \tilde{b}_p^n(p_\ell^{n+1}, w_h)\right| \leqslant C\tau |||w_h|||, \tag{7.4a}$$

$$\left|\tilde{b}_a^{n+1}(s_a^{n+1}, w_h) - \tilde{b}_a^{n}(s_a^{n+1}, w_h)\right| \leqslant C\tau \||w_h\||, \tag{7.4b}$$

$$\left|\tilde{b}_v^{n+1}(s_v^{n+1}, w_h) - \tilde{b}_v^{n}(s_v^{n+1}, w_h)\right| \leqslant C\tau \||w_h\||. \tag{7.4c}$$

Moreover,

$$\left|\tilde{f}_p^{n+1}(w_h) - \tilde{f}_p^{n}(w_h)\right| \leqslant C\tau \||w_h\||, \tag{7.5a}$$

$$\left|\tilde{f}_a^{n+1}(w_h) - \tilde{f}_a^{n}(w_h)\right| \leqslant C\tau \||w_h\||, \tag{7.5b}$$

$$\left|\tilde{f}_v^{n+1}(w_h) - \tilde{f}_v^{n}(w_h)\right| \leqslant C\tau \||w_h\||. \tag{7.5c}$$

Proof. The results can be obtained using the Lipschitz continuity of all the coefficients and the smoothness of p_ℓ, s_a, s_v. □

LEMMA 7.2 For any $0 \leqslant n \leqslant N$, and any $w_h \in X_h$, we have the bounds

$$\left|\tilde{b}_p^{n}(p_\ell^{n+1}, w_h) - b_p^{n}(p_\ell^{n+1}, w_h)\right| \leqslant C\left(h^2 + \|S_{a_h}^n - s_a^n\| + \|S_{v_h}^n - s_v^n\|\right) \||w_h\||, \tag{7.6a}$$

$$\left|\tilde{b}_a^{n}(s_a^{n+1}, w_h) - b_a^{n}(s_a^{n+1}, w_h)\right| \leqslant C\left(h^2 + \|S_{a_h}^n - s_a^n\| + \|S_{v_h}^n - s_v^n\|\right) \||w_h\||, \tag{7.6b}$$

$$\left|\tilde{b}_v^{n}(s_v^{n+1}, w_h) - b_v^{n}(s_v^{n+1}, w_h)\right| \leqslant C\left(h^2 + \|S_{a_h}^n - s_a^n\| + \|S_{v_h}^n - s_v^n\|\right) \||w_h\||. \tag{7.6c}$$

Moreover,

$$\left|\tilde{f}_p^{n}(w_h) - f_p^{n}(w_h)\right| \leqslant C\left(h^2 + \|S_{a_h}^n - s_a^n\| + \|S_{v_h}^n - s_v^n\|\right) \||w_h\||, \tag{7.7a}$$

$$\left|\tilde{f}_a^{n}(w_h) - f_a^{n}(w_h)\right| \leqslant C\left(h + \||e_{p_h}^{n+1}\|| + \|S_{a_h}^n - s_a^n\| + \|S_{v_h}^n - s_v^n\|\right) \||w_h\||, \tag{7.7b}$$

$$\left|\tilde{f}_v^{n}(w_h) - f_v^{n}(w_h)\right| \leqslant C\left(h + \||e_{p_h}^{n+1}\|| + \|S_{a_h}^n - s_a^n\| + \|S_{v_h}^n - s_v^n\|\right) \||w_h\||. \tag{7.7c}$$

Proof. We start by showing (7.6a). Note that using the definition of b_p (4.7), we obtain

$$\begin{aligned}\left|\tilde{b}_p^{n}(p_\ell^{n+1}, w_h) - b_p^{n}(p_\ell^{n+1}, w_h)\right| \leqslant &\sum_{K\in\mathscr{E}_h} \int_K \left|\left(\tilde{\lambda}_t^n - \lambda_t^n\right)\kappa \nabla p_\ell^{n+1} \cdot \nabla w_h\right| \\ &+ \sum_{e\in\Gamma_h} \int_e \left|\left\{ \left(\tilde{\lambda}_t^n - \lambda_t^n\right) \kappa \nabla p_\ell^{n+1} \cdot \boldsymbol{n}_e \right\}\left[w_h\right]\right|. \end{aligned} \tag{7.8}$$

With the Lipschitz continuity assumptions H.2, we can write

$$\left|\tilde{\lambda}_t^n - \lambda_t^n\right| \leqslant C(|S_{v_h}^n - s_v^n| + |S_{a_h}^n - s_a^n|).$$

The first term on the right-hand side of (7.8) is bounded by

$$\sum_{K\in\mathscr{E}_h}\int_K\left|\left(\tilde{\lambda}_t^n-\lambda_t^n\right)\kappa\nabla p_\ell^{n+1}\cdot\nabla w_h\right|\leqslant C\left(\|S_{a_h}^n-s_a^n\|+\|S_{v_h}^n-s_v^n\|\right)|||w_h|||,$$

where we have used the Cauchy–Schwarz inequality, the boundedness of κ and the smoothness of p_ℓ. For the second term on the right-hand side of (7.8), we have

$$\begin{aligned}\sum_{e\in\Gamma_h}\int_e\left|\left\{\left(\tilde{\lambda}_t^n-\lambda_t^n\right)\kappa\nabla p_\ell^{n+1}\cdot\boldsymbol{n}_e\right\}\left[w_h\right]\right|&\leqslant C\sum_{e\in\Gamma_h}\int_e\left\{|S_{v_h}^n-s_v^n|+|S_{a_h}^n-s_a^n|\right\}|[w_h]|\\&\leqslant C\sum_{e\in\Gamma_h}\int_e\left\{|e_{v_h}^n|+|e_{a_h}^n|\right\}|[w_h]|+C\sum_{e\in\Gamma_h}\int_e\left\{|e_{v_\pi}^n|+|e_{a_\pi}^n|\right\}|[w_h]|\\&\leqslant C\sum_{e\in\Gamma_h}h_e^{1/2}\left(\|e_{v_h}^n\|_{L^2(e)}+\|e_{a_h}^n\|_{L^2(e)}\right)h_e^{-1/2}\|[w_h]\|_{L^2(e)}\\&\quad+C\sum_{e\in\Gamma_h}h_e^{1/2}\left(\|e_{v_\pi}^n\|_{L^2(e)}+\|e_{a_\pi}^n\|_{L^2(e)}\right)h_e^{-1/2}\|[w_h]\|_{L^2(e)}.\end{aligned}$$

Using the trace inequality (5.3) and the projection estimates (5.6), we have

$$\sum_{e\in\Gamma_h}\int_e\left|\left\{\left(\tilde{\lambda}_t^n-\lambda_t^n\right)\kappa\nabla p_\ell^{n+1}\cdot\boldsymbol{n}_e\right\}\left[w_h\right]\right|\leqslant C\left(\|e_{v_h}^n\|+\|e_{a_h}^n\|\right)|||w_h|||+Ch^2|||w_h|||.$$

Combining these bounds we obtain

$$\left|\tilde{b}_p^n(p_\ell^{n+1},w_h)-b_p^n(p_\ell^{n+1},w_h)\right|\leqslant C\left(h^2+\|S_{a_h}^n-s_a^n\|+\|S_{v_h}^n-s_v^n\|\right)|||w_h|||.$$

The proofs for (7.6b) and (7.6c) are analogous to that of (7.6a) and therefore not shown here.

Next we show (7.7a). Using the definition of f_p, see (4.8), we obtain

$$\left|f_p^n(w_h)-\tilde{f}_p^n(w_h)\right|=\left|f_p(w_h;P_h^n,S_{a_h}^n,S_{v_h}^n)-f_p(w_h;p_\ell^n,s_a^n,s_v^n)\right|\leqslant|T_1|+|T_2|+|T_3|,$$

with

$$\begin{aligned}T_1=&-\sum_{K\in\mathscr{E}_h}\int_K\left(\lambda_v^n\kappa\nabla p_{c,v}^n-\lambda_a^n\kappa\nabla p_{c,a}^n-\kappa\left(\rho\lambda\right)_t^n\boldsymbol{g}\right)\cdot\nabla w_h\\&+\sum_{K\in\mathscr{E}_h}\int_K\left(\tilde{\lambda}_v^n\kappa\nabla\tilde{p}_{c,v}^n-\tilde{\lambda}_a^n\kappa\nabla\tilde{p}_{c,a}^n-\kappa\left(\tilde{\rho}\tilde{\lambda}\right)_t^n\boldsymbol{g}\right)\cdot\nabla w_h,\end{aligned}\qquad(7.9)$$

$$
\begin{aligned}
T_2 = & \sum_{e\in\Gamma_h}\int_e \left\{\lambda_v^n\kappa\nabla p_{c,v}^n\cdot \boldsymbol{n}_e - \tilde{\lambda}_v^n\kappa\nabla\tilde{p}_{c,v}^n\cdot\boldsymbol{n}_e\right\}\left[w_h\right] \\
& - \sum_{e\in\Gamma_h}\int_e \left\{\lambda_a^n\kappa\nabla p_{c,a}^n\cdot \boldsymbol{n}_e - \tilde{\lambda}_a^n\kappa\nabla\tilde{p}_{c,a}^n\cdot\boldsymbol{n}_e\right\}\left[w_h\right], \\
T_3 = & -\sum_{e\in\Gamma_h}\int_e\left\{\kappa\lambda_t^n\boldsymbol{g}\cdot\boldsymbol{n}_e\right\}\left[w_h\right] + \sum_{e\in\Gamma_h}\int_e\left\{\kappa\tilde{\lambda}_t^n\boldsymbol{g}\cdot\boldsymbol{n}_e\right\}\left[w_h\right].
\end{aligned}
\tag{7.10}
$$

Using the boundedness and growth conditions of $\nabla p_{c,a}$, $\nabla p_{c,v}$ (see H.3), the boundedness of the rest of the coefficients and H.2, the volume terms T_1 are bounded as

$$
|T_1| \leqslant C\left(\|S_{a_h}^n - s_a^n\| + \|S_{v_h}^n - s_v^n\|\right)|||w_h|||.
$$

With the same assumptions, for the face terms T_2 we have

$$
\begin{aligned}
|T_2| \leqslant & \, C\sum_{e\in\Gamma_h}\int_e\Big(|S_{a_h}^n - s_a^n| + |S_{v_h}^n - s_v^n|\Big)|[w_h]| = C\sum_{e\in\Gamma_h}\int_e h_e^{1/2}\Big(|e_{a_h}^n| + |e_{v_h}^n|\Big)h_e^{-1/2}|[w_h]| \\
& + C\sum_{e\in\Gamma_h}\int_e h_e^{1/2}\Big(|e_{a_\pi}^n| + |e_{v_\pi}^n|\Big)h_e^{-1/2}|[w_h]| \leqslant C\left(\|S_{a_h}^n - s_a^n\| + \|S_{v_h}^n - s_v^n\|\right)|||w_h||| + Ch^2|||w_h|||,
\end{aligned}
\tag{7.11}
$$

where we have used the trace inequality (5.3), (7.2) and the projection estimates (5.6). Similarly, the terms in T_3 are bounded by

$$
|T_3| \leqslant C\left(h^2 + \|S_{a_h}^n - s_a^n\| + \|S_{v_h}^n - s_v^n\|\right)|||w_h|||,
$$

which concludes the proof.

To prove (7.7b), note that

$$
\left|f_a^n(w_h) - \tilde{f}_a^n(w_h)\right| = \left|f_a(w_h; P_h^{n+1}, S_{a_h}^n, S_{v_h}^n) - f_a(w_h; p_\ell^{n+1}, s_a^n, s_v^n)\right| \leqslant |T_1| + |T_2| + |T_3|,
$$

with

$$
\begin{aligned}
T_1 = & \sum_{K\in\mathscr{E}_h}\int_K\left(\lambda_a^n\boldsymbol{u}_h^{n+1} + \rho_a\lambda_a^n\boldsymbol{g}\right)\cdot\nabla w_h - \sum_{K\in\mathscr{E}_h}\int_K\left(\tilde{\lambda}_a^n\boldsymbol{u}^{n+1} + \rho_a\tilde{\lambda}_a^n\boldsymbol{g}\right)\cdot\nabla w_h, \\
T_2 = & -\sum_{e\in\Gamma_h}\int_e(\lambda_a^n)_{s_a}^{\uparrow}\,\boldsymbol{u}_h^{n+1}\cdot\boldsymbol{n}_e\left[w_h\right] + \sum_{e\in\Gamma_h}\int_e\tilde{\lambda}_a^n\boldsymbol{u}^{n+1}\cdot\boldsymbol{n}_e\left[w_h\right], \\
T_3 = & -\sum_{e\in\Gamma_h}\int_e\left\{\rho_a\kappa\lambda_a^n\boldsymbol{g}\cdot\boldsymbol{n}_e\right\}\left[w_h\right] + \sum_{e\in\Gamma_h}\int_e\left\{\rho_a\kappa\tilde{\lambda}_a^n\boldsymbol{g}\cdot\boldsymbol{n}_e\right\}\left[w_h\right],
\end{aligned}
$$

where we recall that $\boldsymbol{u}_h^{n+1} = \Pi_{\mathrm{RT}}(-\kappa\nabla P_h^{n+1})$ and we denote $\boldsymbol{u}^{n+1} = -\kappa\nabla p_\ell^{n+1}$. Note that we have

$$|T_1| \leqslant \sum_{K\in\mathscr{E}_h}\int_K \left|\left(\lambda_a^n \boldsymbol{u}_h^{n+1} - \tilde{\lambda}_a^n \boldsymbol{u}^{n+1}\right)\cdot\nabla w_h\right| + \sum_{K\in\mathscr{E}_h}\int_K \left|\left(\rho_a\lambda_a^n \boldsymbol{g} - \rho_a\tilde{\lambda}_a^n\boldsymbol{g}\right)\cdot\nabla w_h\right|.$$

The second term is bounded by

$$\sum_{K\in\mathscr{E}_h}\int_K \left|\left(\rho_a\lambda_a^n \boldsymbol{g} - \rho_a\tilde{\lambda}_a^n\boldsymbol{g}\right)\cdot\nabla w_h\right| \leqslant C\left(\|S_{a_h}^n - s_a^n\| + \|S_{v_h}^n - s_v^n\|\right)|||w_h|||.$$

For the first term in T_1 we have

$$\sum_{K\in\mathscr{E}_h}\int_K \lambda_a^n\left|\left(\boldsymbol{u}_h^{n+1} - \boldsymbol{u}^{n+1}\right)\cdot\nabla w_h\right| + \sum_{K\in\mathscr{E}_h}\int_K \left|(\lambda_a^{n+1} - \tilde{\lambda}_a^{n+1})\boldsymbol{u}^{n+1}\cdot\nabla w_h\right|. \tag{7.12}$$

We write

$$\sum_{K\in\mathscr{E}_h}\int_K \left|\lambda_a^n\left(\boldsymbol{u}_h^{n+1} - \boldsymbol{u}^{n+1}\right)\cdot\nabla w_h\right| \leqslant C\|\boldsymbol{u}_h^{n+1} - \boldsymbol{u}^{n+1}\|\,|||w_h|||.$$

Using the triangle inequality, (5.5) and (5.9) we have

$$\|\boldsymbol{u}_h^{n+1} - \boldsymbol{u}^{n+1}\| \leqslant \|\boldsymbol{u}_h^{n+1} + \kappa\nabla_h P_h^{n+1}\| + \|\kappa\nabla_h(P_h^{n+1} - p_\ell^{n+1})\| \leqslant C|||e_{p_h}^{n+1}||| + Ch.$$

For the second term in (7.12) we have

$$\sum_{K\in\mathscr{E}_h}\int_K \left|(\lambda_a^n - \tilde{\lambda}_a^n)\kappa\nabla p_\ell^{n+1}\cdot\nabla w_h\right| \leqslant C\left(\|S_{a_h}^n - s_a^n\| + \|S_{v_h}^n - s_v^n\|\right)|||w_h|||,$$

where we have used that $p_\ell \in C^0(0,T;H^3(\Omega))$. So combining the bounds above, we obtain

$$|T_1| \leqslant C\left(h + |||e_{p_h}^{n+1}||| + \|S_{a_h}^n - s_a^n\| + \|S_{v_h}^n - s_v^n\|\right)|||w_h|||.$$

The term T_2 can be written as

$$|T_2| \leqslant \sum_{e\in\Gamma_h}\int_e \left|\left(\lambda_a^n\right)_{s_a}^{\uparrow}(\boldsymbol{u}_h^{n+1} - \boldsymbol{u}^{n+1})\cdot\boldsymbol{n}_e[w_h]\right| + \sum_{e\in\Gamma_h}\int_e \left|\left(\left(\lambda_a^n\right)_{s_a}^{\uparrow} - \tilde{\lambda}_a^n\right)\boldsymbol{u}^{n+1}\cdot\boldsymbol{n}_e[w_h]\right|. \tag{7.13}$$

The first term in (7.13) is bounded by

$$\sum_{e\in\Gamma_h}\int_e \left|\left(\lambda_a^{n+1}\right)_{s_a}^{\uparrow}(\boldsymbol{u}_h^{n+1} - \boldsymbol{u}^{n+1})\cdot\boldsymbol{n}_e[w_h]\right| \leqslant C|||w_h|||\left(\sum_{e\in\Gamma_h} h_e\|\boldsymbol{u}_h^{n+1} - \boldsymbol{u}^{n+1}\|_{L^2(e)}^2\right)^{1/2}.$$

We fix a face e and we choose one neighboring element K_e such that $e \subset \partial K_e$:

$$\|\boldsymbol{u}_h^{n+1} - \boldsymbol{u}^{n+1}\|_{L^2(e)} \leqslant \|\boldsymbol{u}_h^{n+1} + \kappa \nabla P_h^{n+1}|_{K_e}\|_{L^2(e)} + \|\kappa \nabla (P_h^{n+1}|_{K_e} - p_\ell^{n+1})\|_{L^2(e)}.$$

Using the trace inequality (5.3) and the Raviart–Thomas projection estimate (5.9), we have

$$\left(\sum_{e\in\Gamma_h} h_e \|\boldsymbol{u}_h^{n+1} + \kappa \nabla P_h^{n+1}|_{K_e}\|_{L^2(e)}^2\right)^{1/2} \leqslant \|\boldsymbol{u}_h^{n+1} + \kappa \nabla_h P_h^{n+1}\| \leqslant C|||e_{p_h}^{n+1}||| + Ch.$$

Adding and subtracting $\pi_{h,\Gamma_{\mathrm{D}}^{p_\ell}} p_\ell^{n+1}$ and using (5.7) and the trace inequality (5.3) yields

$$\left(\sum_{e\in\Gamma_h} h_e \|\nabla (P_h^{n+1}|_{K_e} - p_\ell^{n+1})\|_{L^2(e)}^2\right)^{1/2} \leqslant C|||e_{p_h}^{n+1}||| + Ch.$$

Therefore, the first term in (7.13) is bounded as

$$\sum_{e\in\Gamma_h} \int_e \left| (\lambda_a^n)_{s_a}^{\uparrow} (\boldsymbol{u}_h^{n+1} - \boldsymbol{u}^{n+1}) \cdot \boldsymbol{n}_e [w_h] \right| \leqslant \ C(h + |||e_{p_h}^{n+1}|||)|||w_h|||.$$

The second term in (7.13) can be bounded as

$$\sum_{e\in\Gamma_h} \int_e \left| \left((\lambda_a^n)_{s_a}^{\uparrow} - \tilde{\lambda}_a^n \right) \boldsymbol{u}^{n+1} \cdot \boldsymbol{n}_e [w_h] \right| \leqslant C \left(h^2 + \|S_{a_h}^n - s_a^n\| + \|S_{v_h}^n - s_v^n\| \right) |||w_h|||.$$

Therefore, combining the bounds above and using that $h \leqslant 1$ so that $h^2 \leqslant h$, we have

$$|T_2| \leqslant C \left(h + |||e_{p_h}^{n+1}||| + \|S_{a_h}^n - s_a^n\| + \|S_{v_h}^n - s_v^n\| \right) |||w_h|||.$$

Finally, the term T_3 is bounded by

$$|T_3| \leqslant C \left(h^2 + \|S_{a_h}^n - s_a^n\| + \|S_{v_h}^n - s_v^n\| \right) |||w_h|||.$$

Combining all the bounds above gives the result.

The proof for (7.7c) is analogous to that of (7.7b). □

7.1 *Liquid pressure*

The following lemma gives an equation for the error $e_{p_h}^n$.

LEMMA 7.3 (Error equation for the liquid pressure). We have that, for all $w_h \in X_{h,\Gamma_{\mathrm{D}}^{p_\ell}}$ and all $0 \leqslant n \leqslant N-1$, there exists a constant $C > 0$ independent of h and τ such that

$$b_p^n(e_{p_h}^{n+1}, w_h) \leqslant b_p^n(e_{p_\pi}^{n+1}, w_h) + C\left(\tau + h^2 + \|S_{a_h}^n - s_a^n\| + \|S_{v_h}^n - s_v^n\|\right) |||w_h|||. \tag{7.14}$$

Proof. Subtracting the consistency of the scheme (6.1) from the discretization (4.6), we have

$$b_p^n(P_h^{n+1}, w_h) - \tilde{b}_p^{n+1}(p_\ell^{n+1}, w_h) = f_p^n(w_h) - \tilde{f}_p^{n+1}(w_h). \tag{7.15}$$

This is equivalent to

$$b_p^n(P_h^{n+1}, w_h) - \tilde{b}_p^n(p_\ell^{n+1}, w_h) = f_p^n(w_h) - \tilde{f}_p^n(w_h) + \tilde{f}_p^n(w_h) - \tilde{f}_p^{n+1}(w_h) - \tilde{b}_p^n(p_\ell^{n+1}, w_h) + \tilde{b}_p^{n+1}(p_\ell^{n+1}, w_h). \tag{7.16}$$

Thanks to (7.4a) and (7.5a) we have

$$b_p^n(P_h^{n+1}, w_h) - \tilde{b}_p^n(p_\ell^{n+1}, w_h) = f_p^n(w_h) - \tilde{f}_p^n(w_h) + C\tau |||w_h|||. \tag{7.17}$$

This is also equivalent to

$$b_p^n(P_h^{n+1}, w_h) - b_p^n(p_\ell^{n+1}, w_h) = f_p^n(w_h) - \tilde{f}_p^n(w_h) + C\tau |||w_h||| + \tilde{b}_p^n(p_\ell^{n+1}, w_h) - b_p^n(p_\ell^{n+1}, w_h). \tag{7.18}$$

Owing to (7.6a), (7.7a), this is equivalent to

$$b_p^n(P_h^{n+1} - p_\ell^{n+1}, w_h) \leqslant C\left(\tau + h^2 + \|S_{a_h}^n - s_a^n\| + \|S_{v_h}^n - s_v^n\|\right) |||w_h|||. \tag{7.19}$$

The result is obtained by using (7.2a). □

LEMMA 7.4 (Error estimates for the liquid pressure). We have that, for all $0 \leqslant n \leqslant N-1$,

$$\tilde{C}\tau |||e_{p_h}^{n+1}|||^2 \leqslant C\tau(\tau^2 + h^2) + C\tau\left(\|S_{a_h}^n - s_a^n\|^2 + \|S_{v_h}^n - s_v^n\|^2\right), \tag{7.20}$$

where $C, \tilde{C} > 0$ are independent of h and τ.

Proof. Letting $w_h = \tau e_{p_h}^{n+1}$ in (7.14), it reads

$$\tau b_p^n(e_{p_h}^{n+1}, e_{p_h}^{n+1}) \leqslant \tau b_p^n(e_{p_\pi}^{n+1}, e_{p_h}^{n+1}) + C\tau\left(\tau + h^2 + \|S_{a_h}^n - s_a^n\| + \|S_{v_h}^n - s_v^n\|\right) |||e_{p_h}^{n+1}|||. \tag{7.21}$$

Applying coercivity (6.11) and boundedness (6.19) of b_p^n, we obtain

$$C_{\alpha,p_\ell}\tau |||e_{p_h}^{n+1}|||^2 \leqslant C_{\mathrm{B}*,p_\ell}\tau |||e_{p_\pi}^{n+1}|||_* |||e_{p_h}^{n+1}||| + C\tau\left(\tau + h^2 + \|S_{a_h}^n - s_a^n\| + \|S_{v_h}^n - s_v^n\|\right) |||e_{p_h}^{n+1}|||. \tag{7.22}$$

Using Young's inequality on the right-hand side, we have

$$C\tau \| | e_{p_h}^{n+1} \| |^2 \leqslant C\tau \| | e_{p_\pi}^{n+1} \| |_*^2 + C\tau \left(\tau^2 + h^4 + \| S_{a_h}^n - s_a^n \|^2 + \| S_{v_h}^n - s_v^n \|^2 \right). \tag{7.23}$$

The result is obtained after using the L^2-orthogonal projection estimates (5.8) and recalling that $h^4 \leqslant h^2$. □

7.2 *Aqueous saturation*

The following lemma gives an equation for the error $e_{a_h}^n$.

LEMMA 7.5 (Error equation for the aqueous saturation). We have that, for all $w_h \in X_{h,\Gamma_D^{s_a}}$, and all $0 \leqslant n \leqslant N-1$, there exists a constant $C > 0$ independent of h and τ such that

$$\begin{aligned} \frac{1}{\tau}(\phi(S_{a_h}^{n+1} - s_a^{n+1}), w_h) + b_a^n(e_{a_h}^{n+1}, w_h) &= \frac{1}{\tau}(\phi(S_{a_h}^n - s_a^n), w_h) + b_a^n(e_{a_\pi}^{n+1}, w_h) \\ &+ C\left(\tau + h + \| | e_{p_h}^{n+1} \| | + \| S_{a_h}^n - s_a^n \| + \| S_{v_h}^n - s_v^n \| \right) \| | w_h \| | + \sigma_a(w_h), \end{aligned} \tag{7.24}$$

where

$$\sigma_a(w_h) = \frac{1}{\tau}(\beta_a^{n+1}, w_h), \quad \beta_a^{n+1} = \phi \int_{t_n}^{t_{n+1}} (t - t_n) \partial_{tt} s_a \, \mathrm{d}t. \tag{7.25}$$

Proof. Using a Taylor series expansion we have

$$s_a^n = s_a^{n+1} - \tau \left(\partial_t s_a \right)^{n+1} + \int_{t_n}^{t_{n+1}} \left(t - t_n \right) \partial_{tt} s_a \, \mathrm{d}t, \tag{7.26}$$

Rearranging the terms, multiplying by ϕ and a test function $w_h \in X_h$ and integrating over Ω yields

$$\frac{1}{\tau}(\phi s_a^{n+1}, w_h) = \frac{1}{\tau}(\phi s_a^n, w_h) + (\phi(\partial_t s_a)^{n+1}, w_h) - \sigma_a(w_h). \tag{7.27}$$

where $\sigma_a(w_h)$ is defined in (7.25). Using the consistency of scheme (6.25) on the second term on the right-hand side, rearranging terms and subtracting the result from (4.11) reads

$$\begin{aligned} \frac{1}{\tau}(\phi(S_{a_h}^{n+1} - s_a^{n+1}), w_h) + b_a^n(S_{a_h}^{n+1}, w_h) - \tilde{b}_a^{n+1}(s_a^{n+1}, w_h) &= \frac{1}{\tau}(\phi(S_{a_h}^n - s_a^n), w_h) \\ &+ f_a^n(w_h) - \tilde{f}_a^{n+1}(w_h) + \sigma_a(w_h). \end{aligned} \tag{7.28}$$

Owing to (7.6b) and (7.7b), this is equivalent to

$$\frac{1}{\tau}(\phi(S_{a_h}^{n+1} - s_a^{n+1}), w_h) + b_a^n(S_{a_h}^{n+1} - s_a^{n+1}, w_h) = \frac{1}{\tau}(\phi(S_{a_h}^{n} - s_a^{n}), w_h) + C\left(\tau + h + |||e_{p_h}^{n+1}||| + \|S_{a_h}^n - s_a^n\| + \|S_{v_h}^n - s_v^n\|\right) |||w_h||| + \sigma_a(w_h). \quad (7.29)$$

Using (7.2) gives the result. □

LEMMA 7.6 (Error estimates for the aqueous saturation). We have that, for all $0 \leqslant n \leqslant N-1$,

$$\|S_{a_h}^{n+1} - s_a^{n+1}\|^2 + \tilde{C}\tau |||e_{a_h}^{n+1}|||^2 \leqslant (1 + C\tau)\|S_{a_h}^n - s_a^n\|^2 + C\tau\|S_{v_h}^n - s_v^n\|^2 + C\tau(\tau^2 + h^2), \quad (7.30)$$

where $C, \tilde{C} > 0$ are independent of h and τ.

Proof. Let $w_h = \tau e_{a_h}^{n+1}$ in (7.24):

$$\phi(S_{a_h}^{n+1} - s_a^{n+1}, e_{a_h}^{n+1}) + \tau b_a^n(e_{a_h}^{n+1}, e_{a_h}^{n+1}) = \phi(S_{a_h}^{n} - s_a^{n}, e_{a_h}^{n+1}) + \tau b_a^n(e_{a_\pi}^{n+1}, e_{a_h}^{n+1}) + C\tau\left(\tau + h + |||e_{p_h}^{n+1}||| + \|S_{a_h}^n - s_a^n\|\right) |||e_{a_h}^{n+1}||| + \tau\sigma_a(e_{a_h}^{n+1}). \quad (7.31)$$

Using the coercivity of b_a^n (6.28) on the second term of the left-hand side, and the boundedness of b_a^n (6.32) on the first term of the right-hand side, we obtain

$$\phi(S_{a_h}^{n+1} - s_a^{n+1}, e_{a_h}^{n+1}) + C_{\alpha,s_a}\tau |||e_{a_h}^{n+1}|||^2 \leqslant \phi(S_{a_h}^{n} - s_a^{n}, e_{a_h}^{n+1}) + C_{B*,s_a}\tau |||e_{a_\pi}^{n+1}|||_* |||e_{a_h}^{n+1}||| + C\tau\left(\tau + h + |||e_{p_h}^{n+1}||| + \|S_{a_h}^n - s_a^n\|\right) |||e_{a_h}^{n+1}||| + \tau\sigma_a(e_{a_h}^{n+1}). \quad (7.32)$$

Note that $(S_{a_h}^{n+1} - s_a^{n+1}, e_{a_h}^{n+1}) = \|S_{a_h}^{n+1} - s_a^{n+1}\|^2 + (S_{a_h}^{n+1} - s_a^{n+1}, e_{a_\pi}^{n+1})$ and $(S_{a_h}^{n} - s_a^{n}, e_{a_h}^{n+1}) = (S_{a_h}^{n} - s_a^{n}, S_{a_h}^{n+1} - s_a^{n+1}) + (S_{a_h}^{n} - s_a^{n}, e_{a_\pi}^{n+1})$. Moreover,

$$(S_{a_h}^{n} - s_a^{n}, S_{a_h}^{n+1} - s_a^{n+1}) = \frac{1}{2}\|S_{a_h}^{n+1} - s_a^{n+1}\|^2 + \frac{1}{2}\|S_{a_h}^{n} - s_a^{n}\|^2 - \frac{1}{2}\|S_{a_h}^{n+1} - s_a^{n+1} - S_{a_h}^{n} + s_a^{n}\|^2,$$

where we have used that $ab = \frac{1}{2}a^2 + \frac{1}{2}b^2 - \frac{1}{2}(a-b)^2$. Thus,

$$\frac{\phi}{2}\|S_{a_h}^{n+1} - s_a^{n+1}\|^2 + C_{\alpha,s_a}\tau |||e_{a_h}^{n+1}|||^2 \leqslant \frac{\phi}{2}\|S_{a_h}^{n} - s_a^{n}\|^2 + \phi(S_{a_h}^{n} - S_{a_h}^{n+1}, e_{a_\pi}^{n+1}) + \phi(s_a^{n+1} - s_a^{n}, e_{a_\pi}^{n+1}) + C_{B*,s_a}\tau |||e_{a_\pi}^{n+1}|||_* |||e_{a_h}^{n+1}||| + C\tau\left(\tau + h + |||e_{p_h}^{n+1}||| + \|S_{a_h}^n - s_a^n\|\right) |||e_{a_h}^{n+1}||| + \tau\sigma_a(e_{a_h}^{n+1}). \quad (7.33)$$

The second term on the right-hand side is zero due to (7.3). Moreover, the third term on the right-hand side is bounded above by $C\tau\|e_{a_\pi}^{n+1}\| \leqslant C\tau h^2$. Thus,

$$\frac{\phi}{2}\|S_{a_h}^{n+1} - s_a^{n+1}\|^2 + C_{\alpha,s_a}\tau|||e_{a_h}^{n+1}|||^2 \leqslant \frac{\phi}{2}\|S_{a_h}^{n} - s_a^{n}\|^2 + C_{B*,s_a}\tau|||e_{a_\pi}^{n+1}|||_*|||e_{a_h}^{n+1}||| \\ + C\tau\left(\tau + h + |||e_{p_h}^{n+1}||| + \|S_{a_h}^{n} - s_a^{n}\|\right)|||e_{a_h}^{n+1}||| + \tau\sigma_a(e_{a_h}^{n+1}) + C\tau h^2. \quad (7.34)$$

Note that, using the Cauchy–Schwarz inequality, we have $\tau\sigma_a(e_{a_h}^{n+1}) \leqslant \|\beta_a^{n+1}\|\ \|e_{a_h}^{n+1}\|$. Moreover,

$$\begin{aligned}\|\beta_a^{n+1}\|^2 &= \int_\Omega\left(\phi\int_{t_n}^{t_{n+1}}(t-t_n)\partial_{tt}s_a\,\mathrm{d}t\right)^2 \leqslant C\int_\Omega\left(\int_{t_n}^{t_{n+1}}(t-t_n)\partial_{tt}s_a\,\mathrm{d}t\right)^2\\ &\leqslant C\int_\Omega\left(\int_{t_n}^{t_{n+1}}(t-t_n)^2\,\mathrm{d}t\right)\left(\int_{t_n}^{t_{n+1}}(\partial_{tt}s_a)^2\,\mathrm{d}t\right)\\ &\leqslant C\tau^3\int_\Omega\int_{t_n}^{t_{n+1}}(\partial_{tt}s_a)^2\,\mathrm{d}t\\ &\leqslant C\tau^4\max_{t\in[t_n,t_{n+1}]}\int_\Omega(\partial_{tt}s_a)^2\\ &= C\tau^4\max_{t\in[t_n,t_{n+1}]}\|\partial_{tt}s_a\|^2\\ &\leqslant C\tau^4,\end{aligned}$$

where we have used that $s_a \in C^2(0,T;L^2(\Omega))$. Using this, Young's inequality on the right-hand side and estimates (5.8), we obtain

$$\frac{\phi}{2}\|S_{a_h}^{n+1} - s_a^{n+1}\|^2 + C\tau|||e_{a_h}^{n+1}|||^2 \leqslant \frac{\phi}{2}\|S_{a_h}^{n} - s_a^{n}\|^2 + C\tau\left(\tau^2 + h^2 + |||e_{p_h}^{n+1}|||^2 + \|S_{a_h}^{n} - s_a^{n}\|^2\right). \quad (7.35)$$

Using the liquid pressure error estimates (7.20) gives the result. □

7.3 *Vapor saturation*

The following two lemmas state error estimates for the error $e_{v_h}^n$. Proofs are similar to those of Lemmas 7.5 and 7.6 and thus are skipped for brevity.

LEMMA 7.7 (Error equation for the vapor saturation). We have that, for all $w_h \in X_{h,\Gamma_{\mathrm{D}}^{sv}}$ and all $0 \leqslant n \leqslant N-1$, there exists a constant $C > 0$ independent of h and τ such that

$$\frac{1}{\tau}(\phi(S_{v_h}^{n+1} - s_v^{n+1}), w_h) + b_v^n(e_{v_h}^{n+1}, w_h) = \frac{1}{\tau}(\phi(S_{v_h}^n - s_v^n), w_h) + b_v^n(e_{v_\pi}^{n+1}, w_h)$$
$$+ C\left(\tau + h + |||e_{p_h}^{n+1}||| + \|S_{a_h}^n - s_a^n\| + \|S_{v_h}^n - s_v^n\|\right) |||w_h||| + \sigma_v(w_h), \quad (7.36)$$

where

$$\sigma_v(w_h) = \frac{1}{\tau}(\beta_v^{n+1}, w_h), \quad \beta_v^{n+1} = \phi \int_{t_n}^{t_{n+1}} (t - t_n)\partial_{tt} s_v \, \mathrm{d}t. \quad (7.37)$$

LEMMA 7.8 (Error estimates for the vapor saturation). We have that, for all $0 \leqslant n \leqslant N-1$,

$$\|S_{v_h}^{n+1} - s_v^{n+1}\|^2 + \tilde{C}\tau |||e_{v_h}^{n+1}|||^2 \leqslant (1 + C\tau)\|S_{v_h}^n - s_v^n\|^2 + C\tau\|S_{a_h}^n - s_a^n\|^2 + C\tau(\tau^2 + h^2), \quad (7.38)$$

where $C, \tilde{C} > 0$ are independent of h and τ.

7.4 *Final estimates*

In this section we combine the error estimates (7.20), (7.30) and (7.38), and use induction to give the final error bounds. We first denote the errors made with the starting values by $\mathscr{E}(t_0)$:

$$\mathscr{E}(t_0) = \|S_{a_h}^0 - s_a^0\|^2 + \|S_{v_h}^0 - s_v^0\|^2.$$

THEOREM 7.1 There exists a constant C independent of h and τ such that the following error estimates hold:

$$\|S_{a_h}^N - s_a^N\|^2 + \|S_{v_h}^N - s_v^N\|^2 + C\tau\left(|||e_{p_h}^N|||^2 + |||e_{a_h}^N|||^2 + |||e_{v_h}^N|||^2\right) \leqslant e^{CT}\left(\mathscr{E}(t_0) + C(\tau^2 + h^2)\right). \quad (7.39)$$

Proof. Let $A_{n+1} = \|S_{a_h}^{n+1} - s_a^{n+1}\|^2 + \|S_{v_h}^{n+1} - s_v^{n+1}\|^2$, $B_{n+1} = C\tau\left(|||e_{p_h}^{n+1}|||^2 + |||e_{a_h}^{n+1}|||^2 + |||e_{v_h}^{n+1}|||^2\right)$ and $D = C\tau(\tau^2 + h^2)$. Then, adding up all three estimates (7.20), (7.30) and (7.38), we obtain

$$A_{n+1} + B_{n+1} \leqslant (1 + C\tau)A_n + D.$$

Applying induction, we have that, for any $1 \leqslant n \leqslant N$,

$$A_n + B_n \leqslant (1 + C\tau)^n A_0 + D\sum_{k=0}^{n-1}(1 + C\tau)^k.$$

We apply this with $n = N$. Since $(1 + C\tau)^k \leqslant (1 + C\tau)^N \leqslant e^{CN\tau} = e^{CT}$ we have

$$A_N + B_N \leqslant e^{CT} A_0 + D \sum_{k=0}^{n-1} e^{CT} \leqslant e^{CT} A_0 + (N-1) D e^{CT} \leqslant e^{CT}(A_0 + C(\tau^2 + h^2)),$$

which concludes the proof. □

COROLLARY 7.1 Assume that the initial solutions $S^0_{a_h}, S^0_{v_h}$ satisfy (4.20). Then we have

$$\|S^N_{a_h} - s^N_a\|^2 + \|S^N_{v_h} - s^N_v\|^2 + C\tau \left(|||e^N_{p_h}|||^2 + |||e^N_{a_h}|||^2 + |||e^N_{v_h}|||^2 \right) \leqslant Ce^{CT} \left(\tau^2 + h^2 \right). \tag{7.40}$$

Proof. This follows from (7.39). □

REMARK 7.1 Note that, as expected, the convergence rates in space are suboptimal. As we show in Section 8, by setting $\tau = h^2$, we can recover second-order convergence. In order to obtain optimal rates of convergence in space, a duality argument is needed.

8. Numerical results

For the numerical results, we consider manufactured solutions under different scenarios. The solution of the problem is given by

$$p_\ell(t, x, y) = 2 + xy^2 + x^2 \sin(t + y), \tag{8.1a}$$

$$s_a(t, x, y) = \frac{1 + 2x^2y^2 + \cos(t + x)}{8}, \tag{8.1b}$$

$$s_v(t, x, y) = \frac{3 - \cos(t + x)}{8}. \tag{8.1b}$$

The computational domain is taken as $\Omega = [0, 1] \times [0, 1]$, and the final time of the problem is $T = 1$. The porosity ϕ is taken to be constant equal to 0.2, while the absolute permeability κ is taken to be constant equal to 1. The phase viscosities are set as

$$\mu_\ell = 0.75, \quad \mu_v = 0.25, \quad \mu_a = 0.5. \tag{8.2}$$

The phase relative permeabilities and the capillary pressures are defined as (Bentsen & Anli, 1976; Chen *et al.*, 2006)

$$k_{r\ell} = s_\ell(s_\ell + s_a)(1 - s_a), \quad k_{rv} = s_v^2, \quad k_{ra} = s_a^2, \tag{8.3a}$$

$$p_{c,v} = \frac{3.9}{\ln(0.01)} \ln(1.01 - s_v), \quad p_{c,a} = \frac{6.3}{\ln(0.01)} \ln(s_a + 0.01). \tag{8.3b}$$

We consider Dirichlet boundary conditions on all the boundaries of the domain. The source terms q_ℓ, q_v and q_a are computed according to the manufactured solutions and other parameters of the problem.

TABLE 1 *Rates of convergence for test case in Section 8.1, with $\tau = h$*

h	DOFs	p_ℓ $L^2(\Omega)$-error	Rate	s_a $L^2(\Omega)$-error	Rate	s_v $L^2(\Omega)$-error	Rate
0.25	64	3.18e−2	—	7.41e−3	—	5.84e−2	—
0.125	256	1.14e−2	1.48	4.67e−3	0.66	9.64e−3	2.60
0.0625	1,024	2.78e−3	2.04	2.27e−3	1.04	4.77e−3	1.02
0.03125	4,096	9.22e−4	1.59	1.18e−3	0.94	2.15e−3	1.15
0.015625	16,384	3.41e−4	1.44	6.01e−4	0.97	1.08e−3	1.01

TABLE 2 *Rates of convergence for test case in Section 8.1, with $\tau = h^2$*

h	DOFs	p_ℓ $L^2(\Omega)$-error	Rate	s_a $L^2(\Omega)$-error	Rate	s_v $L^2(\Omega)$-error	Rate
0.5	16	1.36e−1	—	6.48e−3	—	5.11e−2	—
0.25	64	3.40e−2	2.00	1.51e−3	2.10	3.37e−3	3.92
0.125	256	8.43e−3	2.01	3.74e−4	2.01	6.95e−4	2.28
0.0625	1,024	2.11e−3	2.00	9.35e−5	2.00	1.85e−4	1.91
0.03125	4,096	5.32e−4	1.99	2.32e−5	2.01	5.07e−5	1.87

TABLE 3 *Rates of convergence for test case in Section 8.2, with $\tau = h$*

h	DOFs	p_ℓ $L^2(\Omega)$-error	Rate	s_a $L^2(\Omega)$-error	Rate	s_v $L^2(\Omega)$-error	Rate
0.25	64	3.20e−2	—	8.10e−3	—	6.05e−2	—
0.125	256	1.20e−2	1.42	5.06e−3	0.68	1.11e−2	2.45
0.0625	1,024	2.78e−3	2.11	2.42e−3	1.06	5.03e−3	1.14
0.03125	4,096	9.78e−4	1.51	1.27e−3	0.93	2.08e−3	1.27
0.015625	16,384	3.66e−4	1.42	6.47e−4	0.97	1.04e−3	1.00

8.1 *Constant densities*

First, we consider the case in which the phase densities are constant and taken as

$$\rho_\ell = 3, \quad \rho_v = 1, \quad \rho_a = 5. \tag{8.4}$$

For this test case, gravity is not considered. We take $\theta_{p_\ell} = \theta_{s_a} = \theta_{s_v} = 1$ and $\alpha_{p_\ell,e} = \alpha_{s_a,e} = \alpha_{s_v,e} = 1$ on all the edges of the mesh. The simulation is performed on six uniform meshes with an initial mesh size of $h = 0.5$. We compute the L^2-errors at the final time. In Tables 1 and 2, we show the results with the time step τ taken equal to h and to h^2, respectively. We observe that, as expected from the results in Section 7, when $\tau = h$, the scheme is first order. Moreover, we can recover second order when taking $\tau = h^2$.

8.2 *Gravity*

Finally, we consider the effect of gravity. We take $\boldsymbol{g} = [0 \ -0.1]^{\mathrm{T}}$, and the phase densities as taken as in (8.4). The simulation is performed on six uniform meshes with an initial mesh size of $h = 0.5$.

TABLE 4 *Rates of convergence for test case in Section 8.2, with $\tau = h^2$*

h	DOFs	p_ℓ $L^2(\Omega)$-error	Rate	s_a $L^2(\Omega)$-error	Rate	s_v $L^2(\Omega)$-error	Rate
0.5	16	1.36e−1	—	6.53e−3	—	5.50e−2	—
0.25	64	3.43e−2	1.99	1.56e−3	2.07	3.72e−3	3.89
0.125	256	8.47e−3	2.02	3.79e−4	2.04	6.55−4	2.51
0.0625	1,024	2.13e−3	1.99	9.51e−5	1.99	1.81e−4	1.86
0.03125	4,096	5.35e−4	1.99	2.37e−5	2.00	5.03e−5	1.85

We compute the L^2-errors at the final time. In Tables 3 and 4, we show the results with the time step τ taken equal to h and to h^2, respectively. We observe that, as expected from the results in Section 7, when $\tau = h$, the scheme is first order. Moreover, we can recover second order when taking $\tau = h^2$.

9. Conclusions

We presented and analyzed a first-order discontinuous Galerkin method for the incompressible three-phase flow problem in porous media. Our method does not require a subiteration scheme, which makes it computationally cheaper. We obtained *a priori* error estimates by assuming Lipschitz continuity of the coefficients. The numerical test cases show, under different scenarios, that our scheme is first-order convergent. For future work, we would like to extend the numerical analysis to variable density flow and to a second-order scheme by using a BDF2 time stepping. Moreover, we plan to extend this scheme to the black oil problem where mass transfer can occur between the liquid and vapor phases and study the performance of such methods on setups that include wells or viscous fingering effects (Bangerth *et al.*, 2006; Li & Rivière, 2016).

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