Supplementary material

Numerical method. The incompressible Navier-Stokes equations with variable density ρ and dynamical viscosity η are formulated as follows:

$$\partial_t \rho + \boldsymbol{\nabla}\cdot(\rho \boldsymbol{u}) = 0,$$
$$\partial_t \boldsymbol{m} + \boldsymbol{\nabla}(\boldsymbol{m}\otimes \boldsymbol{u}) - 2\boldsymbol{\nabla}\cdot(\eta\varepsilon(\boldsymbol{u})) + \boldsymbol{\nabla} p = \boldsymbol{f},$$
$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0,$$

where $\varepsilon(\boldsymbol{u}) := \frac{1}{2}(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^{\mathsf{T}})$, $\boldsymbol{m} := \rho\boldsymbol{u}$ and $\boldsymbol{f}$ is a forcing term. To approximate the above system, we developed a method similar to the one proposed in [14]. We use the momentum $\boldsymbol{m}$ as dependent variable and we rewrite the diffusion operator $\boldsymbol{\nabla}\cdot(\bar{\nu}\varepsilon(\boldsymbol{m}))+\boldsymbol{\nabla}\cdot(-\bar{\nu}\varepsilon(\boldsymbol{m})+\eta\varepsilon(\boldsymbol{u}))$ with $\bar{\nu} \geq \eta/\rho$. The first term is treated implicitly while the second term is made explicit so the algorithm is suitable for spectral methods. The main novelty consists of enforcing the incompressibility condition using an artificial compression method and not a pressure-correction projection method as we did in [14]. As a consequence, a term $\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{u})$ appears in the momentum equation which is treated like the diffusion operator. More precisely, it is rewritten $\boldsymbol{\nabla}(\frac{1}{\bar{\rho}}\boldsymbol{\nabla}\cdot\boldsymbol{m}) + \boldsymbol{\nabla}(-\frac{1}{\bar{\rho}}\boldsymbol{\nabla}\cdot\boldsymbol{m} + \boldsymbol{\nabla}\cdot\boldsymbol{u})$ with $\bar{\rho} \leq \rho$. The first term is treated implicitly while the second term is made explicit.

Convergence study. We perform two sets of tests that use the immiscible fluid pair Ga-Hg to show the convergence properties of the above numerical method. We recall that the physical properties of the fluids are summarized in Table I of the article. All the tests reported in the following are performed on a cell of size $R = H_1 = H_2 = 2\,\text{cm}$.

The first series of tests consists of studying a hydrodynamic setting where we initialize the problem with a small amplitude gravity wave $(m, n) = (1, 1)$. It allows us to compare the frequency of oscillations ω and the viscous damping λ_{visc} we obtain numerically with the theoretical values obtained using the linear theory described in [8]. The tests are performed on three uniform grids of typical mesh size $h \in \{10^{-3}, 5\times10^{-4}, 2.5\times10^{-4}\}$ using various time steps $\delta t \in \{2, 1, 0.5, 0.25\}$ ms and a final time $T = 5\,\text{s}$. The results are summarized in Table I and show that the algorithm converges. Figure 1 (left) displays the time evolution of the kinetic energy of the azimuthal Fourier mode $m = 1$ for various meshes. The viscous damping λ_{visc} is computed by using a linear fit between the maxima of the kinetic energy of the Fourier mode $m = 1$ over each oscillation. The relative error in viscous damping remains in the range $3\% - 15\%$ even for the time step $\delta t = 0.25\,\text{ms}$. This leads us to consider a smaller time step for the following tests.

mesh size h in mm	0.5				0.25				0.125			
time step δt in ms	2	1	0.5	0.25	2	1	0.5	0.25	2	1	0.5	0.25
ω in s^{-1}	15.3	16.7	17.4	17.8	15.3	16.8	17.6	18.0	15.4	16.8	17.6	18.1
Relative Error e_ω	14.8E-2	7.0E-2	3.1E-2	0.8E-2	14.8E-2	6.4E-2	1.9E-2	0.3E-2	14.2E-2	6.4E-2	1.9E-2	0.8E-2
λ_{visc} in s^{-1}	-0.226	-0.164	-0.131	-0.118	-0.208	-0.161	-0.130	-0.113	-0.198	-0.158	-0.139	-0.126
Relative Error $e_{\lambda_{\text{visc}}}$	1.05	4.9E-1	1.9E-1	7.3E-2	8.9E-1	4.6E-1	1.8E-1	2.7E-2	8.0E-1	4.4E-1	2.6E-1	1.5E-1

TABLE I: Convergence tests on hydrodynamic setting using 16 real Fourier modes. The theoretical frequency is $\omega_{th} = 17.95\,\text{s}^{-1}$ and the theoretical viscous damping is $\lambda_{\text{visc}} = -0.110\,\text{s}^{-1}$.

The second series of tests focuses on the magnetohydrodynamic (MHD) set up described in the section "Numerical simulations". We show here that using the time step $\delta t = 0.1\,\text{ms}$ in the numerical simulations allows us to approximate correctly the dynamics of the problem. We recall that to capture the viscous damping, we need to resolve viscous boundary layers as thin as 0.08 mm. This leads us to perform tests on a nonuniform grid with a mesh size $h = 0.5\,\text{mm}$ near the top/bottom of the cell ($z = \pm 2\,\text{cm}$) where the flow magnitude is small and $h = 0.075\,\text{mm}$ near the plane $z = 0$. We run three sets of tests with $J \in \{1.5, 2, 3\} \times 10^5\,\text{Am}^{-2}$, $B_z = 15\,\text{mT}$, time step $\delta t \in \{0.25, 0.1\}$ ms and the final time $T = 5\,\text{s}$. The results are summarized in Table II and show that the magnetohydrodynamic simulations are well resolved when using the time step $\delta t = 0.1\text{ms}$ and a nonuniform mesh size $h = 0.5 - 0.075\,\text{mm}$. The evolution of the kinetic energy of the Fourier mode $m = 1$ for $J = 3\times 10^5\,\text{Am}^{-2}$ is displayed in semi-log scale in Figure 1 (right). It is used to compute the growth rate λ via linear fit.

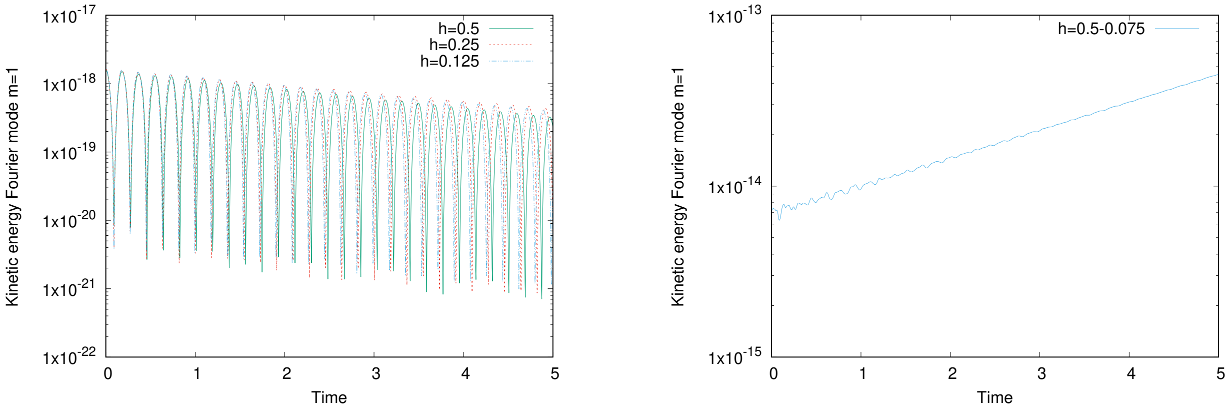

FIG. 1: Time evolution of the kinetic energy associated to the Fourier mode $m = 1$. (left) Hydrodynamic setting for various mesh sizes h in mm and the time step $\delta t = 0.25$ms. (right) MHD setting with $J = 3\times10^5\,\mathrm{Am}^{-2}$, $B_z = 15\,\mathrm{mT}$, $h = 0.5-0.075\,\mathrm{mm}$ and $\delta t = 10^{-4}s$. The energy is plotted in semi-log scale.

J in $10^5\,\mathrm{Am}^{-2}$	1.5		2		3	
time step δt in ms	0.25	0.1	0.25	0.1	0.25	0.1
λ in s^{-1}	0.0325	0.0436	0.0845	0.0958	0.1800	0.2003
Relative Error e_λ	1.9E-1	8.0E-2	9.5E-2	2.6E-2	9.8E-2	3.7E-3

TABLE II: Convergence tests on magnetohydrodynamic setting using 32 real Fourier modes for various values of J with $B_z = 15$mT. The theoretical growth rate λ is respectively $\{0.040, 0.093, 0.200\}$ for J in $\{1.5, 2, 3\}10^5\,\mathrm{Am}^{-2}$. Nonuniform mesh size with $h = 0.5\,\mathrm{mm}$ near $z = \pm2\,\mathrm{cm}$ and $h = 0.075\,\mathrm{mm}$ near $z = 0$.